AN ITERATIVE METHOD FOR FINDING THE TRANSIENT RESPONSE OF AN AUTOMATICALLY CONTROLLED MISSILE HAVING NON-LINEAR STIFFNESS

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6 June 1960

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SUMMARY

An equation of missile motion in the pitching plane was found, assuming a non-linearity in aerodynamic stiffness in the form of a cubic term. An iterative method for approximating the angle-of-attack response to a step displacement of elevator angle was formulated. The results of applications of this method were compared with computer solutions; convergence of the approximations WAS ~~were~~ demonstrated.

A simple feedback control system was designed, and the iterative method was extended to the case of a control system step input.

The method was thought to have limited practical value because of the labor required in using it.

CONTENTS

Tables

Figures

Appendices

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INTRODUCTION

Important non-linearities which are often present in low aspect ratio configurations are those of the lift versus incidence and the pitching moment versus incidence characteristics. In missile design these non-linearities, as well as any others present, are likely to be ignored initially, simply because linear system design is well developed and relatively easy. Automatic computing techniques may then be used to determine the behaviour of the system when the non-linearities are included.

It would be a great help for the designer to be able to find the transient response of a contemplated design without going to the computer. It was for this reason that the present investigation was undertaken. By expressing the pitching moment as the sum of a linear term and a cubic term in angle-of-attack, ignoring the non-linearity in lift because it is likely to be less important, an equation amenable to iterative solution can be formulated. The first approximation to the transient response may be expressed as a Fourier series, which must be cubed to perform the first iteration. The main effort of the development will be to formulate a practical method for performing the cubing process, for the first and subsequent iterations, and to systematise the steps necessary for carrying out the iterations.

First, an equation relating angle-of-attack to elevator angle will be derived, and the response to a step displacement of elevator

angle will be found. Later, a simple automatic feedback control system will be designed for the purpose of formulating a method of finding the response to a control system input.

The author takes this opportunity to express his gratitude for the generous assistance given by Mr. P. A. T. Christopher, who instigated and supervised this investigation.

ENGLISH-LETTER NOMENCLATURE

a_o	coefficient of restoring term in Sec. 3
a_1	coefficient of damping term in Sec. 3
a_{ijk}	coefficients of terms in the cube of Fourier series in Sec. 2
A_n	gain of frequency response at frequency $n\omega_F$
b	coefficient of cubic term in Sec. 3
b_n	coefficients in the Fourier series for α_{1v}
B_n	phase shift of frequency response at frequency $n\omega_F$
c_n	coefficients in Sec. 2
C_n	phase angles in the Fourier series for α_{1v}
C(s)	$1/K_2$ times the transfer function of the compensator-amplifier
C_L	lift coefficient
C_M	pitching moment coefficient
C_1	partial derivative of M_V with respect to α_v
C_2	coefficient of vane damping term
db	decibels
D	d/dt
D	missile body diameter
E	actuating error
E_n	coefficients in trigonometric identities
F_n	phase angles in trigonometric identities

$H(s)$	transfer function used in closed-loop iteration
i	$\sqrt{(-1)}$
I	moment of inertia of missile about y axis
I_V	moment of inertia of vane about its c.g.
K_1	zero frequency gain of the aerodynamic transfer function
K_2	zero frequency gain of compensator-amplifier
K_3	gain of input amplifier
l_V	distance between missile c.g. and vane c.g.
L	lift
m	mass of missile
M	aerodynamic pitching moment on missile about y axis
M_V	aerodynamic pitching moment on vane about its c.g.
$M'(w)$	component of moment due to w
$M''(\alpha)$	component of moment due to α
M_α	coefficient of the linear term in the cubic expression for pitching moment
M_{α^3}	coefficient of the cubic term in the cubic expression for pitching moment
M_η	partial derivative of M with respect to η
n	harmonic
$P(s)$	$1/K_1$ times aerodynamic transfer function
q	angular velocity of missile about the y axis
Q	step function in Sec. 3

Q	stagnation pressure in Appendices I and VI
$\lvert Q \rvert$	magnitude of step function
Q_o	non-dimensional step input
$\lvert Q_o \rvert$	magnitude of non-dimensional step input
R	ratio of non-linear to linear steady-state values
s	Laplace transform variable
S	wing area
t	time
t_1	time measured from start of second half cycle
T	period of square wave
$T(D)$	differential operator in equation for missile aerodynamics
u	component of missile c.g. velocity in x direction
$U(D)$	differential operator in equation for missile aerodynamics
V	velocity of missile
$V(s)$	vane transfer function
w	component of missile c.g. velocity in z direction
$W(s)$	actuator transfer function
Z	aerodynamic force component in the z direction
Z_w	partial derivative of Z with respect to w
Z_α	partial derivative of Z with respect to α
Z_η	partial derivative of Z with respect to η

GREEK-LETTER NOMENCLATURE

α	missile angle-of-attack
α_D	demanded missile angle-of-attack
$\lvert\alpha_D\rvert$	magnitude of input step function
α_R	reference input
α_V	vane angle-of-attack
$\alpha_1, \alpha_2, \ldots$	approximations to α
$\alpha_3^*, \alpha_4^* \ldots$	corrections to approximations
α_{1_U}	response of linear system to unit step input
β	coefficient of the cubic term
γ	angle between missile axis and vane axis
ζ	damping ratio
ζ_1	damping ratio of missile
ζ_2	damping ratio of vane
ζ_3	damping ratio of elevator actuator
η	elevator deflection angle
$\lvert\eta\rvert$	magnitude of step elevator input
$\lvert\eta_D\rvert$	demanded elevator angle
θ_M	angle between missile axis and space reference
θ_V	angle between vane axis and space reference
τ	time constant in the missile aerodynamics transfer function
ω_F	fundamental frequency of square wave
ω_n	natural frequency of linear system

ω_{n_1} natural frequency of linear missile

ω_{n_2} natural frequency of vane

ω_{n_3} natural frequency of elevator actuator

SUBSCRIPTS NOT SHOWN ELSEWHERE

a evaluated function

b function to be derived from an evaluated function

L linear

N non-linear

S steady-state

SECTION 1: THE EQUATION OF MISSILE MOTION IN PITCH

The system of axes and sign conventions to be used are illustrated in Fig. 1. Body axes are used. The missile to be considered has a fixed wing and an all-moving tail. It may or may not be of cruciform configuration, since pure pitching motion will be assumed.

The rigid body equations of motion in the z direction and about the y axis (which passes through the center of mass) are, respectively,

$$Z = m(\dot{w} - qu) \tag{1-1}$$

and

$$M = I\dot{q} \tag{1-2}$$

The same equations would, of course, arise if the general three dimensional equations of motion were written down and the confining assumptions of the present motion imposed.

The component in the z direction of the total force on the missile is taken as

$$Z = Z_w w + Z_\eta \eta \tag{1-3}$$

The simplifying assumption has been made here that the gravitational force may be neglected because it is small compared with the aerodynamic forces available. The contribution to Z due to q is omitted on the assumption that it is small. Its inclusion would not change the nature of the analysis, but it would be dropped eventually anyway because values for it, for the configuration and flight conditions considered, are not available. Also implicit in Eq. 1-3 is the assumption that the malalignment force is zero, i.e. Z is zero when w and η are zero.

The aerodynamic moment is taken as

$$M = M'(w) + M_\eta \eta \tag{1-4}$$

where $M'(w)$ is to be read, "The component of moment due to w." This is the non-linear term. The same remarks as those following Eq. 1-3 apply to the exclusion of components of moment due to q and to malalignment.

Now we take

$$u = V \tag{1-5}$$

which is valid for small angles of attack, and we assume that V is constant. Then, for small angles of attack,

$$\alpha = \frac{w}{V} \tag{1-6}$$

and

$$Z_w = \frac{Z_\alpha}{V} \tag{1-7}$$

Substituting Eqs. 1-5 and 1-6 in Eq. 1-1 gives

$$Z = m(V\dot{\alpha} - qV) \tag{1-8}$$

Substituting Eqs. 1-6 and 1-7 in Eq. 1-3 gives

$$Z = Z_\alpha \alpha + Z_\eta \eta \tag{1-9}$$

Also we replace $M'(w)$ by $M''(\alpha)$, where $M''(\alpha)$ is to be read, "The component of moment due to α." Thus Eq. 1-4 becomes

$$M = M''(\alpha) + M_\eta \eta \tag{1-10}$$

Combining Eqs. 1-8 and 1-9 gives

$$Z_\alpha \alpha + Z_\eta \eta = m(V\dot{\alpha} - qV) \tag{1-11}$$

Combining Eqs. 1-2 and 1-10 gives

$$M''(\alpha) + M_\eta \eta = I\dot{q} \tag{1-12}$$

Eliminating q between Eqs. 1-11 and 1-12 gives

$$mIV\ddot{\alpha} - IZ_{\alpha}\dot{\alpha} - MVM''(\alpha) = IZ_{\eta}\dot{\eta} + mVM_{\eta}\eta \tag{1-13}$$

Taking

$$M''(\alpha) = M_{\alpha}\alpha + M_{\alpha^3}\alpha^3 \tag{1-14}$$

equation 1-13 becomes

$$mIV\ddot{\alpha} - IZ_{\alpha}\dot{\alpha} - mVM_{\alpha}\alpha - mVM_{\alpha^3}\alpha^3 = IZ_{\eta}\dot{\eta} + mVM_{\eta}\eta \tag{1-15}$$

Dividing Eq. 1-15 through by mIV gives

$$\ddot{\alpha} - \frac{Z_{\alpha}}{mV}\dot{\alpha} - \frac{M_{\alpha}}{I}\alpha - \frac{M_{\alpha^3}}{I}\alpha^3 = \frac{Z_{\eta}}{mV}\dot{\eta} + \frac{M_{\eta}}{I}\eta \tag{1-16}$$

The constants ω_{n_1}, ζ_1, K_1, τ and β are now defined as follows:

$$\omega_{n_1}^2 = -\frac{M_{\alpha}}{I} \tag{1-17}$$

$$\zeta_1 = -\frac{Z_{\alpha}}{2mV\omega_{n_1}} \tag{1-18}$$

$$K_1 = \frac{M_{\eta}}{I\omega_{n_1}^2} \tag{1-19}$$

$$\tau = \frac{Z_{\eta}}{mVK_1\omega_{n_1}^2} \tag{1-20}$$

$$\beta = -\frac{M_{\alpha^3}}{I} \tag{1-21}$$

Making these substitutions in Eq. 1-16 gives

$$\ddot{\alpha} + 2\zeta_1\omega_{n_1}\dot{\alpha} + \omega_{n_1}^2\alpha + \beta\alpha^3 = K_1\omega_{n_1}^2(\eta + \tau\dot{\eta}) \tag{1-22}$$

In Ref. 1, E. G. Brown-Edwards calculated the aerodynamic coefficients for a typical missile configuration. This missile is sketched in Fig. 2. In Appendix I Brown Edwards' results are given and are used as the basis of the calculation of the constants in Eq. 1-22. The body diameter was taken as one foot and the flight Mach number as three. The calculations for a flight altitude of

50,000' are shown in the appendix. The results are tabulated in Table I. This table also contains the results for sea level flight, the calculations for which are not shown.

SECTION 2: THE TRANSIENT RESPONSE OF THE MISSILE TO A STEP DISPLACEMENT OF ELEVATOR ANGLE

Equation 1-22 relates the angle-of-attack α to the elevator angle η. We wish to find the variation of α with time upon the application of a step displacement in η, initial values all being zero. No analytical solution for this non-linear problem exists; an iterative method will be sought.

Equation 1-22 is, in operational form,

$$(D^2 + 2\zeta_1 \omega_{n_1} D + \omega_{n_1}^2)\alpha + \beta\alpha^3 = K_1 \omega_{n_1}^2 (1 + \tau D)\eta \tag{2-1}$$

This may be written:

$$T(D)\alpha + \beta\alpha^3 = U(D)\eta \tag{2-2}$$

where

$$T(D) = D^2 + 2\zeta_1 \omega_{n_1} D + \omega_{n_1}^2 \tag{2-3}$$

$$U(D) = K_1 \omega_{n_1}^2 (1 + \tau D) \tag{2-4}$$

Now an approximation to the solution of Eq. 2-2 is the solution $\alpha_1(t)$ of

$$T(D)\alpha_1 = U(D)\eta \tag{2-5}$$

This equation is linear, and could be solved analytically, however we will use an approximate method. In Ref. 1, where an attempt is made to solve the same non-linear problem, Eq. 2-5 <u>is</u> solved analytically. When this is done the ensuing steps become quite tedious. We hope to find an advantage in taking another course.

We find $\alpha_1(t)$ in the form of a Fourier series using the approximate method given by Wass and Haymen in Ref. 2 and described in Appendix II of this thesis. The method requires that the frequency response of Eq. 2-5 be determined. The reader who is unfamiliar with frequency response technique will find it explained in Section 5, in connection with the design of the control system. The Wass, Hayman method gives $\alpha_1(t)$,

approximately, in the form,

$$\alpha_1(t) = |\eta| \left[\frac{A_o}{2} + \frac{2}{\pi} \sum \frac{A_n}{n} \sin\left(n\omega_F t + B_n\right) \right] \qquad (2\text{-}6)$$

$$n = 1, 3, \ldots, 11.$$

where ω_F is a known constant, $|\eta|$ is the magnitude of the step input, and the A's and B's are known constants - gains and phase shifts of the frequency response. If positive damping exists, this response is a damped oscillation, with a steady-state value of

$$(\alpha_1)_S = |\eta| A_o \qquad (2\text{-}7)$$

The true steady-state value $(\alpha)_S$ may be calculated from Eq. 1-22 by setting all the terms involving derivatives equal to zero:

$$\omega_{n_1}^2 (\alpha)_S + \beta (\alpha)_S^3 = K_1 \omega_{n_1}^2 |\eta| \qquad (2\text{-}8)$$

The solution for $(\alpha)_S$ is easy, though not analytic.

Now if we take, as a new approximation to $\alpha(t)$,

$$\alpha_2(t) = \frac{(\alpha)_S}{|\eta| A_o} \alpha_1(t)$$

$$= \frac{(\alpha)_S}{A_o} \left[\frac{A_o}{2} + \frac{2}{\pi} \sum \cdots \right] \qquad (2\text{-}9)$$

then its steady-state value is

$$(\alpha_2)_S = \frac{(\alpha)_S}{A_o} A_o = (\alpha)_S$$

Since $\alpha_2(t)$ has the true steady-state value, there is some reason for thinking it is an improved approximation for $\alpha(t)$. Some further justification is presented in Fig. 3. The curves in this figure represent the solutions, by digital computer, of an equation essentially the same as (1-22). The equation is discussed

in detail in Section 3. The solutions for the linear case and for several non-zero values of the coefficient in the cubic term are plotted. In relation to the present example, the linear solution corresponds to $\alpha_1(t)$, and the non-linear solutions to $\alpha(t)$. In Fig. 4, each of the non-linear solutions has been scaled up to give steady-state values equal to $(\alpha_1)_S$ - an inversion of the process represented by Eq. 2-9. It may be seen that this scaling process apparently brings the non-linear solutions into closer agreement with the linear one in a general sense, not just the steady-state values. Thus $\alpha_2(t)$, a scaled-down version of $\alpha_1(t)$, will be taken as an improved approximation to $\alpha(t)$. For brevity let

$$R = \frac{(\alpha)_S}{|\eta| A_o} \tag{2-10}$$

so that

$$\alpha_2(t) = R\,\alpha_1(t) \tag{2-11}$$

For later convenience let

$$\alpha_{1_\upsilon}(t) = \frac{A_o}{2} + \frac{2}{\pi}\sum \frac{A_n}{n} \sin\,(n\,\omega_F t + B_n) \tag{2-12}$$

i.e. the response of the linear system to a unit step input. Thus,

$$\alpha_1(t) = |\eta|\,\alpha_{1_\upsilon}(t) \tag{2-13}$$

and

$$\alpha_2(t) = R\,|\eta|\,\alpha_{1_\upsilon}(t) \tag{2-14}$$

Now if β is sufficiently small, a closer approximation to $\alpha(t)$ is the solution $\alpha_3(t)$ of

$$\begin{aligned} T(D)\,\alpha_3 &= U(D)\,\eta - \beta\,\alpha_2^3 \\ &= U(D)\,\eta - \beta\,R^3\,|\eta|^3\,\alpha_1^3 \end{aligned} \tag{2-15}$$

Or, if $\alpha_{3*}(t)$ is the solution of

$$T(D)\alpha_{3*} = -\beta\alpha_2^3 = -\beta R^3 \; |\eta|^3 \; \alpha_{1_u}^3 \tag{2-16}$$

then by the principle of linear superposition,

$$\alpha_3(t) = \alpha_1(t) + \alpha_{3*}(t) \tag{2-17}$$

To solve (2-16), a method will be developed for expressing $\alpha_{1_u}^3$ as a Fourier series.

$$\alpha_{1_u}^3 = \left[\frac{A_o}{2} + \frac{2}{\pi} \sum \frac{A_n}{n} \sin (n\omega_F t + B_n) \right]^3$$

$$n = 1, 3, \ldots, 11.$$

The expanded cubic consists of the cube of each of the original terms, plus three times the product of the square of each term and each of the other terms, plus six times the product of all combinations of three unlike terms. Showing a few terms of the expansion and introducing a notation for their coefficients we have:

$$\begin{aligned}
\alpha_{1_u}^3 = {} & a_{000} A_o^3 + a_{001} A_o^2 A_1 \sin (\omega_F t + B_1) + \ldots \\
& + a_{111} A_1^3 \sin^3 (\omega_F t + B_1) \\
& + a_{113} A_1^2 A_3 \sin^2 (\omega_F t + B_1) \sin (3\omega_F t + B_3) + \ldots \\
& + a_{ijk} A_i A_j A_k \sin (i\omega_F t + B_i) \sin (j\omega_F t + B_j) \sin (k\omega_F t + B_k)
\end{aligned} \tag{2-18}$$

Each term is identified by a subscript i, j, k. The coefficients a_{ijk} take account of the kind of term (e.g. the multiplier 6 for a term which is the product of three unlike terms) and the factors

1/2, $2/\pi$, and 1/n. The coefficients a_{ijk} are tabulated in Table II.

Any product of sine terms can be expressed in the form,

$$E_o + E_1 \sin(\omega_F t + F_1) + E_2 \sin(2\omega_F t + F_2)$$

$$+ \ldots + E_r \sin(r\omega_F t + F_r) + \ldots$$

For example, in the 1,3,5 term we have the identity:

$$\sin(\omega_F t + B_1) \sin(3\omega_F t + B_3) \sin(5\omega_F t + B_5)$$

$$= -\tfrac{1}{4} \sin(\omega_F t - B_1 - B_3 + B_5) + \tfrac{1}{4} \sin(3\omega_F t + B_1 - B_3 + B_5)$$

$$+ \tfrac{1}{4} \sin(7\omega_F t - B_1 + B_3 + B_5) - \tfrac{1}{4} \sin(9\omega_F t + B_1 + B_3 + B_5)$$

In this case

$$E_1 = -\tfrac{1}{4}, \quad E_3 = \tfrac{1}{4}, \quad E_7 = \tfrac{1}{4}, \quad E_9 = -\tfrac{1}{4}$$

$$E_o = E_2 = E_4 = E_5 = E_6 = E_8 = E_{10} = \ldots = 0$$

$$F_1 = -B_1 - B_3 + B_5$$

$$F_3 = B_1 - B_3 + B_5$$

$$F_7 = -B_1 + B_3 + B_5$$

$$F_9 = B_1 + B_3 + B_5$$

Using this notation a large number of these identities may be tabulated concisely, and this has been done, for the combinations necessary here, in Table III.

We adopt the notation:

$$\sin(i\omega_F t + B_i)\sin(j\omega_F t + B_j)\sin(k\omega_F t + B_k)$$

$$= E_{o(ijk)} + E_{1(ijk)}\sin(\omega_F t + F_{1(ijk)}) + \ldots \qquad (2\text{-}19)$$

Also, we will write, for later convenience,

$$\sin(i\omega_F t + B_i)\sin(j\omega_F t + B_j)$$

$$= E_{o(oij)} + E_{1(oij)}\sin(\omega_F t + F_{1(oij)}) + \ldots \qquad (2\text{-}20)$$

and

$$\sin(i\omega_F t + B_i)$$

$$= E_{o(ooi)} + E_{1(ooi)}\sin(\omega_F t + F_{1(ooi)}) + \ldots \qquad (2\text{-}21)$$

The latter expression is a trivial use of such a notation since the right hand side reduces to $\sin(i\omega_F t + B_i)$, but its use helps provide a systematic tabulation.

Using the identities represented by Eqs. 2-19, 2-20, and 2-21 in Eq. 2-18 gives ${\alpha_{1_v}}^3$ as a series of constants and simple sine terms. Since sine terms of like frequency can be combined into a single sine term of that frequency, we have:

$${\alpha_{1_v}}^3 = b_o + b_1\sin(\omega_F t + C_1) + b_2\sin(2\omega_F t + C_2) \qquad (2\text{-}22)$$

$$+ \ldots + b_n\sin(n\omega_F t + C_n) + \ldots$$

where

$$b_o = a_{000} A_o^3 + a_{001} A_o^2 A_1 + \dots$$

$$+ a_{ijk} E_{o(ijk)} A_i A_j A_k + \dots$$

$$b_1 \sin(\omega_F t + C_1) = a_{001} E_{1(001)} A_o^2 A_1 \sin(\omega_F t + F_{1(001)})$$

$$+ a_{003} E_{1(003)} A_o^2 A_3 \sin(\omega_F t + F_{1(003)}) + \dots$$

$$+ a_{ijk} E_{1(ijk)} A_i A_j A_k \sin(\omega_F t + F_{1(ijk)}) + \dots$$

$$b_n \sin(n\,\omega_F t + C_n) = \dots$$

$$+ a_{ijk} E_{n(ijk)} A_i A_j A_k \sin(n\,\omega_F t + F_{n(ijk)}) + \dots$$

It is possible to shorten the notation conveniently by letting

$$c_{n(ijk)} = a_{ijk} E_{n(ijk)} \tag{2-23}$$

The values of $c_{n(ijk)}$ may be determined from the tabulations of a_{ijk} and $E_{n(ijk)}$. Accordingly, values of $c_{n(ijk)}$ (or simply c) are tabulated in Table IV. The coefficients are all listed as positive. The coefficients for which the values of E were negative were changed in sign by adding 180^o to the corresponding angles F, which are also tabulated. It will be seen that like harmonics have been grouped together.

Table IV does not include coefficients for all possible terms. The elimination of less important terms was based on the effect of each term on $\alpha_3(t)$ (the approximation towards which we are working) in the numerical example which follows. It had been hoped, at the outset of this investigation, that the number of terms needed would be small. But it was found necessary to retain a large number of

terms, up to and including 9th harmonic terms. The relative importance of the smallest terms retained can best be seen in the example of this section.

The fact that the requirements of a particular numerical example must be injected at this point is a deficiency in the method, since the choice of retained terms is not necessarily the best choice for other examples to which this method will be applied. However, to follow a more general course would be impractical.

To obtain values for b_n and C_n for a particular problem, the products $c_{n(ijk)}\ A_i\ A_j\ A_k$ and the angles $F_{n(ijk)}$ are calculated using the values of gain A and phase shift B from the frequency response curves, then like harmonics are combined. The combination of several sine terms in the same frequency may be done most easily by graphical vector addition. This is described in Appendix III.

The Fourier series for $\alpha_{1\upsilon}^{3}$ may be computed with minimum effort by using the form of Table IV. All the necessary results of the preceding development are summarised there, and the required computations are indicated by its layout. To use the table, proceed as follows:

1. Enter the values of gain A and phase shift B of the appropriate frequency response. Once this is done, no further reference need be made to any information other than that contained in Table IV.

2. Calculate the products $A_i\ A_j\ A_k$.

3. Calculate the products $c_{n(ijk)}\ A_i\ A_j\ A_k$. (In the table cAAA is written for $c_{n(ijk)}\ A_i\ A_j\ A_k$)

4. Calculate the angles $F_{n(ijk)}$.

5. Combine like harmonics (e.g. by the method of Appendix III), and enter the results, b_n and C_n, in the space provided.

Step 5 above gives the Fourier series for $\alpha_{1_\upsilon}^{\ 3}$. Multiplying each of the coefficients of the series by the constant $- R^3 |\eta|^3$ gives the Fourier series for $- R^3 |\eta|^3 \alpha_{1_\upsilon}^{\ 3}$. By Eq. 2-16, $\alpha_{3*}(t)$ may be found by applying the frequency response $\beta/T(i\omega)$.

The next approximation to $\alpha(t)$, if it is desired to go further, is $\alpha_4(t)$, the solution of

$$T(D)\alpha_4 = U(D)\,\eta - \beta\alpha_3^{\ 3} \tag{2-24}$$

The determination of a Fourier series for $\alpha_3^{\ 3}$ may be obviated as follows. Adding and subtracting $\beta\alpha_2^{\ 3}$ on the right hand side of (2-24):

$$T(D)\alpha_4 = U(D)\,\eta - \beta\alpha_2^{\ 3} + \beta\,(\alpha_2^{\ 3} - \alpha_3^{\ 3}) \tag{2-25}$$

Let $\alpha_{4*}(t)$ be the solution of

$$T(D)\,\alpha_{4*} = \beta\,(\alpha_2^{\ 3} - \alpha_3^{\ 3}) \tag{2-26}$$

Then by superposition of the solutions of Eqs. 2-15 and 2-26,

$$\alpha_4(t) = \alpha_3(t) + \alpha_{4*}(t) \tag{2-27}$$

The Fourier series for $(\alpha_2^{\ 3} - \alpha_3^{\ 3})$ need not be determined with as great an accuracy as would be required for that of $\alpha_3^{\ 3}$. To solve Eq. 2-26, α_2 and α_3 are each evaluated and cubed, point by point. The difference is then plotted, and a numerical harmonic

analysis is performed. The frequency response $\beta/T(i\omega)$ is applied to the resulting Fourier series, giving α_{4*}. Instead of evaluating $\alpha_2(t)$ and cubing each point, the series which has been obtained for α_2^3 could be evaluated. the former method is preferable, if α_2 has been evaluated, because cubing a few numbers is much easier than evaluating a Fourier series. But the latter method has a certain superiority in that if the series obtained for α_2^3 is not a good representation of the true cube of α_2, due either to a mistake or to a deficiency in the cubing method, then the error will be compensated only if the latter method is used. A deficiency in the cubing method would probably exist if the linear system frequency response was very different from that used in formulating the cubing method.

Improved approximations, α_5, α_6, etc., would be found in the same way as is α_4.

A method is shown in Section 6, in connection with the control system response, for simplifying the calculations when the responses to more than one input magnitude are required. An analogous method can and should be applied here if more than one input magnitude is used.

We now apply this method for the case of missile flight at sea level and an input magnitude of .25 radian. This case was chosen because its analog computer solution was available. The constants in Eqs. 2-2, 2-3, and 2-4 are those tabulated for sea level in Table I, with one exception: τ is taken as zero. The effect of that term on the transient response was found to be negligible, having an influence on the frequency response only at very high frequencies, and it was omitted from the analog computer setup. It was retained, in the development just shown, for generality.

The frequency response of Eq. 2-5,

$$\frac{U}{T}(i\omega) = \frac{K_1 \omega_{n_1}^2}{(i\omega)^2 + 2\zeta_1 \omega_{n_1}(i\omega) + \omega_{n_1}^2}$$

$$= \frac{K_1}{1 - \left(\frac{\omega}{\omega_{n_1}}\right)^2 + 2\zeta_1 \frac{\omega}{\omega_{n_1}} i} \tag{2-28}$$

is plotted in Fig. 5. Gains are plotted in decibels, and phase shifts in degrees. Following the Wass, Hayman method for determining linear transient response, we take ω_F to be one fifth of the frequency at peak gain. The peak occurs at

$$\omega = \omega_{n_1}$$

So that

$$\omega_F = .2\,\omega_{n_1}$$

The gains A_n and phase shifts B_n of Eq. 2-28, at frequencies $n\omega_F$ where $n = 0, 1, 2, \ldots, 13$, are tabulated in Columns 2 and 3 of Table V, with values of n in Column 1. The values of gain are absolute, not decibels. This procedure is necessary to make the computations systematic (this differs slightly from the Wass, Hayman procedure).

The coefficients of the sine terms in the Fourier series for a square wave, magnitude .25, are tabulated in Column 4 of Table V. Each coefficient is identified with the harmonic to which it applies by the value of n in the same row, e.g. the coefficient of the $\sin 7\omega_F t$ term, or seventh harmonic, is .0228. This procedure will be followed throughout the thesis, including the use of a second column for phase angle when required, and will be referred to as "tabulating the Fourier series."

Having tabulated the Fourier series for the input square wave, and the frequency response of the linear system, we find the coefficients of the Fourier series for the output, $\alpha_1(t)$, by multiplying each input coefficient by the corresponding gain. The phase angles of the output series are, in this case, the phase shifts of the frequency response. The Fourier series for $\alpha_1(t)$ is tabulated in Columns 5 and 6. From this tabulation we see, for instance, that the fifth harmonic term of the Fourier series for $\alpha_1(t)$ is $.0775 \sin (5\omega_F t - 88.1^\circ)$.

The evaluation of $\alpha_1(t)$, for 39 values of time up to 95 percent of the half-period of the square wave, is tabulated in Table VI. The evaluation was made, as were all of those in this report, using the Wass, Hayman method. $\alpha_1(t)$ is plotted in Fig. 6 together with the corresponding solution obtained by Brown-Edwards on a Short Analog Computer (see Ref. 1). The oscillation is seen to be of high amplitude, due to the low damping. It is perhaps surprising that the approximation is as good as it is, because the Wass, Hayman method assumes that the oscillation will be nearly completely damped out by the end of the half cycle of the square wave, and it clearly is not.

Solving Eq. 2-8 for $|\eta| = .25$ we find

$$(\alpha)_S = .195$$

By Eq. 2-10,

$$R = \frac{.195}{(.25)(.853)} = .917$$

By Eq. 2-11, the evaluation of $\alpha_2(t)$ is found from the evaluation of $\alpha_1(t)$. This is tabulated in Table VI, and plotted, with the analog computer solution of the non-linear equation, in Fig. 7. $\alpha_{1_U}^3$ is calculated by the method of Table IV. This calculation is shown in Table VII, and the resulting Fourier series has been transferred to Columns 7 and 8 of Table V. Multiplying the magnitudes by

$$R^3 |\eta|^3 = (.917)^3 (.25)^3 = .01206$$

and adding 180° to the phase angles gives the series for $- R^3 |\eta|^3 \alpha_{1_U}^3$, or $- \alpha_2^3$, and this is tabulated in Columns 9 and 10. The frequency response $\beta/T(i\omega)$, which in this case only differs from $U(i\omega)/T(i\omega)$ by a constant multiplier, is tabulated in Columns 11 and 12. By Eq. 2-16, α_{3*} is found by applying the frequency response $\beta/T(i\omega)$ to the Fourier series for $-\alpha_2^3$, and the resulting Fourier series is tabulated in Columns 13 and 14. This is evaluated for 27 values of time in Table VI, and, by Eq. 2-17, combined with $\alpha_1(t)$ to give $\alpha_3(t)$. $\alpha_3(t)$ is plotted in Fig. 7.

In Fig. 7 we see the relationship between the "reduced" linear solution $\alpha_2(t)$, the result of the first iteration $\alpha_3(t)$, and the computer solution $\alpha(t)$. Superficially α_3 doesn't look much better than α_2, particularly with respect to the amplitudes of the peaks and troughs. The <u>times</u> at which the overshoots occur have been brought into line however, and it will be shown in Section 3 that the next iteration gives rapid convergence. The solution to the present example was not taken further because the digital computer

solution of a similar equation became available to the author, and the analog computer solution was thought not to have sufficient accuracy to justify its use as a standard with which to compare more refined approximations.

SECTION 3: COMPARISON OF RESULTS WITH SOLUTION OBTAINED WITH A DIGITAL COMPUTER

Mr. P. A. T. Christopher obtained solutions, from a Ferranti Mercury digital computer, of the equation:

$$D^2x + a_1Dx + a_ox + bx^3 = Q \tag{3-1}$$

where $a_1 = .6$,
$a_o = 1$,
Q is a step function,
Initial values are zero.

These solutions were obtained for various values of b and magnitudes of Q. The accuracy of the solutions was such that, for present purposes, they may be considered exact.

The output $x(t)$, of Eq. 3-1, is dependent on four parameters: a_1, a_o, b, and $|Q|$, the magnitude of the step input. It is shown in Appendix IV that by linear transformations of the variables x and t, the output can be made a function of only two parameters: the damping ratio ζ, and the non-dimensional input magnitude $|Q_o|$. Thus the solution of Eq. 1-22, which is of the same form when $\tau = 0$, could be found from the solution to Eq. 3-1, if ζ and $|Q_o|$ for each of the equations were identical. Now the solutions to Eq. 3-1 include a range of values of $|Q_o|$, but only one value of ζ, i.e. .3, while the value of ζ for the missile at sea level is .175. Accepting this deficiency, we will apply the approximation method to Eq. 3-1 for the computer value of $|Q_o|$ nearest that of

the case considered in Section 2. The results will <u>not</u> be transformed to the amplitude and time scales of the missile problem, because having already seen the approximate missile solution in Section 2, we are now only interested in the convergence of the approximations - which does not depend upon scale.

The equivalent symbols in Eqs. 3-1 and 1-22 are tabulated below ($\zeta = 0$):

Eq. 3-1	Eq. 1-22
a_1	$2\zeta_1\omega_{n_1}$
a_o	$\omega_{n_1}^2$
b	β
Q	$K_1\omega_{n_1}^2\eta$

Thus the non-dimensional input in terms relating to Eq. 1-22 is:

$$Q_o = \frac{Q\sqrt{b}}{a_o^{3/2}} = \frac{(K_1\omega_{n_1}^2\eta)\sqrt{\beta}}{(\omega_{n_1}^2)^{3/2}} = \frac{K_1\sqrt{\beta}\,\eta}{\omega_{n_1}}$$

The value of $|Q_o|$ corresponding to the case considered in Section 2, where the constants are those given in Table I for sea level flight and $|\eta| = .25$, is:

$$|Q_o| = \frac{(.853)\sqrt{(1346)(.25)}}{(24.4)} = .325$$

The computer solutions nearest this are:

(1)
$$b = .5$$
$$|Q| = .4$$
$$|Q_o| = \frac{.4\sqrt{.5}}{(1)^{3/2}} = .283$$

(2) $b = 1$

$|Q| = .4$

$$|Q_o| = \frac{.4\sqrt{1}}{(1)^{3/2}} = .400$$

We choose the higher value, case (2), and proceed with the application of the approximation method to this case. In order to use the equations of Section 2 unaltered, we put Eq. 3-1 in the form of Eq. 2-2 by letting

$$\alpha = x$$
$$\beta = b = 1$$
$$T(D) = D^2 + a_1 D + a_o = D^2 + .6D + 1$$
$$U(D) = a_o = 1$$
$$\eta = \frac{Q}{a_o} = Q$$
$$|\eta| = |Q| = .4$$

The gain of the frequency response

$$\frac{U(i\omega)}{T(i\omega)} = \frac{1}{1 - \omega^2 + .6\,\omega i} \tag{3-2}$$

has a peak at $\omega = 1$, so we take $\omega_F = .2$. Values of gain and phase shift of (3-2) are tabulated in Columns 2 and 3 of Table VIII, for the required values of $n\omega_F$. The Fourier series for a square wave, magnitude .4, is tabulated in Column 4. By Eq. 2-5 we find the Fourier series for $\alpha_1(t)$, and this is tabulated in Columns 5 and 6. The evaluation of $\alpha_1(t)$ for 22 values of time is tabulated in Table IX, and is plotted in Fig. 8 with the corresponding computer solution. The calculated solution α_1 appears to be a good approximation to the true output of the linear system.

The steady-state output is found from Eq. 2-2 by setting the derivative terms equal to zero:

$$(\alpha)_S + (\alpha)_S^3 = |\eta|$$

For $|\eta| = .4$ this equation yields

$$(\alpha)_S = .355$$

By Eq. 2-10,

$$R = \frac{.355}{(.4)\ (1)} = .888$$

The evaluation of $\alpha_2(t)$ is found from the evaluation of $\alpha_1(t)$ according to Eq. 2-11. This is tabulated in Table IX, and is plotted in Fig. 9.

The calculation of $\alpha_{1_U}^3$ is performed in Table X. The resulting Fourier series is entered in Columns 7 and 8 of Table VIII. Multiplying the magnitudes by

$$R^3 |\eta|^3 = (.888)^3\ (.4)^3 = .0447$$

and adding 180° to the phase angles gives the series for $-R^3|\eta|^3\ \alpha_{1_U}^3$ or $-\alpha_2^3$, and this is tabulated in Columns 9 and 10. The frequency response $\beta/T(i\omega)$ is the same as $U(i\omega)/T(i\omega)$ in this case because $\beta = U(i\omega) = 1$. This has already been tabulated in Columns 2 and 3. Thus α_{3*} is found according to Eq. 2-16, tabulated in Columns 11 and 12, and evaluated in Table IX. α_3 is obtained using Eq. 2-17, and plotted in Fig. 9.

The next approximation requires a numerical harmonic analysis. In Table IX, α_2 and α_3 are cubed, and the difference $(\alpha_2^3 - \alpha_3^3)$ is found for each of the 22 values of time. $(\alpha_2^3 - \alpha_3^3)$ is plotted in Fig. 10.

Now harmonic analysis requires that the function being analysed be known throughout one period of the fundamental frequency ω_F. So far only the first half-cycles have been evaluated. The evaluations of α_2 and α_3 in the second half-cycle are given, for 22 values of time, in Table XI. In this tabulation, the time t_1 is the time measured from the start of the second half-cycle. α_2^3, α_3^3, and $(\alpha_2^3 - \alpha_3^3)$ are also tabulated. We see that the values of $(\alpha_2^3 - \alpha_3^3)$ in the second half-cycle are negligible compared with those in the first half-cycle, and set them all equal to zero in the numerical harmonic analysis. This result is not surprising, because the oscillation in the second half-cycle takes place about the steady-state value of zero. Therefore the effect of the non-linearity is small, and the oscillation of the non-linear system is nearly the same as that of the linear one. This can probably be taken as a general result, so that $(\alpha_2^3 - \alpha_3^3)$ need not be evaluated for the second half-cycle.

A 24 ordinate harmonic analysis of $(\alpha_2^3 - \alpha_3^3)$, performed using the layout recommended by Wylie in Ref. 3, is shown in Table XII. The positions at which the ordinates were taken is shown by the vertical lines in Fig. 10. They were found by dividing the first half-cycle into 12 equal time intervals. The ordinates are labeled y_1, y_2, etc., in accordance with Wylie's notation. The ordinates were all multiplied by 1000 to make the harmonic analysis more manageable. The result of the analysis is a series in both sine and cosine terms which includes the 12th harmonic. Retaining only up to the 9th harmonic, the series was put into the form containing only sine terms, and this is tabulated in Columns 13 and 14 of Table VIII. α_{4*} is obtained by Eq. 2-26, and this is tabulated in Columns 15 and 16. It is evaluated in Table IX and, in accordance with Eq. 2-27, added to α_3 to give α_4. This is plotted in Fig. 9 with the other solutions. We

see that the convergence is virtually complete in the region of the first peak, and probably good enough for most practical purposes in the entire range of the computer solution.

This completes the demonstration of the method developed in Section 2. It would be desirable to find out how many iterations are necessary for larger values of input magnitude, and for what values of input magnitude the first iteration is sufficient, but this could not be done by the present author in the time available.

SECTION 4: EQUATIONS OF COMPONENTS OF THE MISSILE CONTROL SYSTEM

A. Incidence Vane

One of the components of the control system, which will be designed in Section 5, is an incidence measuring vane at the nose of the missile. The design of a conical type vane suitable for this use is shown in Appendix V. The vane will be pivoted at its c.g. position.

The relationship between the angle of attack of the missile and the output of the vane must be determined. Fig. 11 shows the angles and sign conventions used in the analysis. The angle γ is measured by a transducer in the vane and is the output of the vane.

It will be seen from the figure that no distinction is made between the velocity of the c.g. of the missile and the velocity of the vane. This is not strictly correct, of course, because the pitching velocity of the missile q creates a component of vane velocity perpendicular to the missile axis with magnitude $l_V\, q$. It is easily shown, however, that for the configuration and flight conditions under consideration, this component of vane velocity has negligible effect on the vane velocity. For the example being considered, under steady-state conditions at an angle of attack of 40°, the component of vane velocity due to pitching of the missile

is 5.3 ft/sec., and the difference between vane incidence and missile incidence is .12°.

The equation of angular motion of the vane about its c.g. is:

$$M_V = I_V \ddot{\theta}_V \tag{4-1}$$

The aerodynamic moment on the vane is taken as

$$M_V = C_1 \alpha_V + C_2 \dot{\gamma} \tag{4-2}$$

where

$$C_1 = \frac{\partial M_V}{\partial \alpha_V}$$

and $C_2 \dot{\gamma}$ is the internal damping moment arising from the relative angular motion between the vane and the missile. It is assumed that a device within the vane exists to provide this damping.

As in the case of the missile (see remarks following Eq. 1-3) the aerodynamic damping moment, that due to $\dot{\alpha}_V$, is omitted on the assumption that it is small, but for the real reason that a value for it is not available.

Combining Eqs. 4-1 and 4-2 gives:

$$C_1 \alpha_V + C_2 \dot{\gamma} = I_V \ddot{\theta}_V \tag{4-3}$$

We wish to reduce Eq. 4-3 to a linear equation in the two variables α and γ. From the figure,

$$\alpha_V = \alpha - \gamma \tag{4-4}$$

and

$$\theta_V = \theta_M - \gamma \tag{4-5}$$

Substituting (4-4) and (4-5) in (4-3) gives

$$C_1 (\alpha - \gamma) + C_2 \dot{\gamma} = I_V (\ddot{\theta}_M - \ddot{\gamma}) \qquad (4\text{-}6)$$

Now $\ddot{\theta}_M$ or $\dot{q}$ represents the pitching accel~~l~~eration of the missile, and $\ddot{\gamma}$ that of the vane relative to the missile. It is reasonable to assume, and it will be evident later, that the frequency of oscillation of the vane is much higher than that of the missile. This being the case, and providing that the magnitude of the missile oscillation is not excessive, then $\ddot{\theta}_M$ may be assumed negligible in comparison with $\ddot{\gamma}$. Accordingly, Eq. 4-6 becomes

$$C_1 (\alpha - \gamma) + C_2 \dot{\gamma} = - I \ddot{\gamma}$$

or

$$\ddot{\gamma} + \frac{C_2}{I_V} \dot{\gamma} - \frac{C_1}{I_V} \gamma = - \frac{C_1}{I_V} \alpha \qquad (4\text{-}7)$$

The constants ω_{n_2} and ζ_2 are now defined as follows:

$$\omega_{n_2}^2 = - C_1 / I_V$$

$$\zeta_2 = C_2 / 2 I_V \omega_{n_2}$$

Eq. 4-7 then becomes

$$\ddot{\gamma} + 2 \zeta_2 \omega_{n_2} \dot{\gamma} + \omega_{n_2}^2 \gamma = \omega_{n_2}^2 \alpha \qquad (4\text{-}8)$$

The calculation of the constants will be found in Appendix VI. They are tabulated in Table I.

B. Actuator

The equation for a typical elevator actuator and the constants of that equation were obtained from D. Morton, a student at the College of Aeronautics who had investigated the subject. The equation is:

$$\ddot{\eta} + 2\zeta_3 \omega_{n_3} \dot{\eta} + \omega_{n_3}^2 \eta = \omega_{n_3}^2 \eta_D \qquad (4\text{-}9)$$

where η_D is the demanded elevator angle,

η is the elevator angle.

The constants ω_{n_3} and ζ_3 are given in Table I.

SECTION 5: DESIGN OF THE CLOSED-LOOP CONTROL SYSTEM

A. The Linear System in Transfer Function Form

The control system arrangement to be used is shown in Fig. 12 in block diagram form. In this diagram:

α_D is the demanded missile angle of attack, the control system input,

α_R is the reference input, equal to K_3 times α_D where the constant K_3 is the gain of the linear amplifier,

α is the missile incidence,

η_D is the demanded elevator angle,

η is the elevator angle,

γ is the output of the incidence measuring vane,

and

$$E = \alpha_R - \gamma = K_3 \alpha_D - \gamma \qquad (5\text{-}1)$$

Equation 4-8 relates α and γ, Eq. 4-9 relates η and η_D,

and Eq. 1-22 relates α and η. Dropping the non-linearity for the present so that the design problem is that of a linear system, Eq. 1-22 becomes

$$\ddot{\alpha} + 2\zeta_1 \omega_{n_1} \dot{\alpha} + \omega_{n_1}^2 \alpha = K_1 \omega_{n_1}^2 (\eta + \tau \dot{\eta}) \qquad (5\text{-}2)$$

The relation between E and η_D is, as yet, unspecified.

The Laplace transformation, defined by:

$$\bar{f}(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

where $\bar{f}(s)$ is the Laplace transform of $f(t)$, will now be applied to Eqs. 4-8, 4-9, 5-1, and 5-2, assuming that initial values of all the variables and their derivatives are zero. Taking the Laplace transforms of these expressions - when initial values are zero - simply amounts to substituting s for d/dt and s^2 for d^2/dt^2. These equations then become, after rearranging,

$$\frac{\bar{\gamma}(s)}{\bar{\alpha}(s)} \equiv V(s) = \frac{\omega_{n_2}^2}{s^2 + 2\zeta_2 \omega_{n_2} s + \omega_{n_2}^2} \qquad (5\text{-}3)$$

$$\frac{\bar{\eta}(s)}{\bar{\eta}_D(s)} \equiv W(s) = \frac{\omega_{n_3}^2}{s^2 + 2\zeta_3 \omega_{n_3} s + \omega_{n_3}^2} \qquad (5\text{-}4)$$

$$\bar{E}(s) = K_3\, \bar{\alpha}_D(s) - \bar{\gamma}(s) \qquad (5\text{-}5)$$

$$\frac{\bar{\alpha}(s)}{\bar{\eta}(s)} \equiv K_1 P(s) = \frac{K_1 \omega_{n_1}^2 (1 + \tau s)}{s^2 + 2\zeta_1 \omega_{n_1} s + \omega_{n_1}^2} \qquad (5\text{-}6)$$

where $V(s)$, $W(s)$, and $P(s)$ are defined as indicated. The transform of the equation relating E and η_D is given the form:

$$\frac{\bar{\eta}_D(s)}{\bar{E}(s)} = K_2 \; C(s) \qquad (5\text{-}7)$$

where

$$C(0) = 1$$

In servomechanism terminology, $V(s)$ is the transfer function of the vane. Similarly $W(s)$, $K_1 \, P(s)$, and $K_2 \, C(s)$ are the transfer functions of the actuator, the missile aerodynamics, and the compensating network-amplifier.

Following standard servomechanism technique, the control system may be represented by the block diagram in Fig. 13.

From Eqs. 5-3, 5-4, 5-6, and 5-7 (or directly from Fig. 13):

$$\frac{\bar{\gamma}}{\bar{E}}(s) = K_1 \, K_2 \, C(s) \;\; W(s) \;\; P(s) \;\; V(s) \qquad (5\text{-}8)$$

This is known as the open-loop transfer function, and the system comprised of the four elements whose transfer functions appear in it and which has an input $E(t)$ and an output $\gamma(t)$ is the "open-loop system".

Combining Eqs. 5-3, 5-5, and 5-8 to eliminate $E(s)$ and $\gamma(s)$ gives the closed-loop transfer function:

$$\frac{\bar{\alpha}}{\bar{\alpha}_D}(s) = \frac{K_1 \, K_2 \, K_3 \, C(s) \, W(s) \, P(s)}{1 + K_1 \, K_2 \, C(s) \, W(s) \, P(s) \, V(s)} \qquad (5\text{-}9)$$

$C(s)$ has not been specified. It has been included in order to provide the system with satisfactory stability characteristics. The stability of the system and the required form for $C(s)$ will

be determined using frequency response technique. The choice of K_2 and K_3 will first be discussed.

B. Steady-State Accuracy

Suppose that α_D is set at some steady value $(\alpha_D)_S$. The rules of Laplace transforms tell us that (in this type of system) α takes on a steady value $(\alpha)_S$ after the transient motion has disappeared, and the ratio $(\alpha)_S/(\alpha_D)_S$ is given by setting $s = 0$ in $\bar{\alpha}(s)/\bar{\alpha}_D(s)$. Accordingly, from Eq. 5-9:

$$\frac{(\alpha)_S}{(\alpha_D)_S} = \frac{\bar{\alpha}}{\bar{\alpha}_D}(0) = \frac{K_1 K_2 K_3}{1 + K_1 K_2} \tag{5-10}$$

since

$$C(0) = W(0) = P(0) = V(0) = 1$$

Ideally the ratio of steady-state values should be one. That is the steady-state output should equal the steady-state demand. We see from Eq. 5-10 that this can be accomplished by a suitable choice of K_3, regardless of the value of $K_1 K_2$. But there is another important consideration: if the missile were to operate entirely at one flight condition, that is at a particular weight, c.g. position, speed, and altitude, then K_1 would always have the same value. But since missiles rarely operate under such conditions, we may assume that K_1 will not maintain a single value throughout the flight, and that $(\alpha)_S/(\alpha_D)_S$ will therefore vary (unless K_3 is made to automatically compensate it). It is evident in Eq. 5-10 that the higher the value of $K_1 K_2$ can be made, the less will be the variation in $(\alpha)_S/(\alpha_D)_S$ for a given change in K_1. K_1 is determined by the aerodynamics of the missile, but K_2 may be chosen.

Aside from considerations of amplifier design, there are two

factors which impose an upper limit on $K_1 K_2$. One is that as $K_1 K_2$ is increased, the system becomes less stable, and finally unstable. So that the choice of K_2 must be a compromise between stability and steady-state accuracy. The other limit imposed upon K_2 is due to the presence of spurious inputs, or noise, which must not be amplified excessively.

When K_2 has been chosen, K_3 may be chosen to give

$$\frac{(\alpha)_S}{(\alpha_D)_S} = \frac{K_1 K_2 K_3}{1 + K_1 K_2} = 1$$

for a given flight condition.

If the steady-state equation for the aerodynamics, including the non-linear term, is combined with the steady-state equations for the other system components to eliminate all the variables except $(\alpha)_S$ and $(\alpha_D)_S$, the following equation is obtained:

$$(\alpha)_S + \frac{\beta}{\omega_{n_1}^2 (1 + K_1 K_2)} (\alpha)_S^3 = \frac{K_1 K_2 K_3}{1 + K_1 K_2} (\alpha_D)_S \tag{5-11}$$

We see that the larger $K_1 K_2$ is made, the smaller is the coefficient of the cubic term. Thus a high open loop gain, seen to be desirable in the linear system, reduces the effective non-linearity in the non-linear system.

C. The Frequency Response of the Linear System

The frequency response of a linear system at the frequency ω is defined as the complex number obtained by substituting $i\omega$ for s in the transfer function relating the output to the input. It is usually expressed in polar form, with the magnitude in units of decibels. A number is expressed in decibels by multiplying its common logarithm by 20.

Now in a linear system a steady sinusoidal input of constant frequency ω will give rise to a steady sinusoidal output with the same frequency, after the transient motion has disappeared. The rules of Laplace transforms tell us that the ratio of the output sinusoidal wave amplitude to the input sinusoidal wave amplitude, or gain, is equal to the <u>magnitude</u> of the frequency response of the system at frequency ω; and that the angle by which the output sinusoidal wave leads the input sinusoidal wave, or phase shift, is equal to the <u>argument</u> of the frequency response at frequency ω. Furthermore, by virtue of the properties of complex numbers, it can be shown that if the transfer function of a system is the product of two or more individual transfer functions, then the phase shift of the system is equal to the sum of the phase shifts of the individual transfer functions; and, if the gains are expressed in decibels, then the gain of the system is equal to the sum of the gains of the individual transfer functions.

The frequency response curves of a system, sometimes called simply the frequency response, are graphs of the system gain and phase shift versus ω (or ω times some constant).

It will be seen that the frequency response of a system may be found by substituting $i\omega$ for D in the differential equation of the system, since the transfer function is found by substituting s for D and the frequency response by substituting $i\omega$ for s. This procedure is used elsewhere in this thesis.

As is shown in textbooks on servomechanisms (see Bibliography), the frequency response curves of the open-loop system determine the stability of the closed-loop system, and they may be used as the basis of trial-and-error design.

Evaluating the individual frequency response curves and combining them in accordance with the rules previously stated,

$P(i\omega)\ V(i\omega)\ W(i\omega)$ was found and is plotted in Fig. 14. The parameter ω/ω_{n_1} is used as the abscissa, and is plotted on a logarithmic scale for convenience. The constants are those given in Table I for flight at 50,000'. It can be seen from Eq. 5-8 that $P(i\omega)\ V(i\omega)\ W(i\omega)$ need only be multiplied by $K_1\ K_2\ C(i\omega)$ to give the open-loop frequency response. It is now necessary to choose a particular compensating network.

In ordinary servo design the choice of the compensator is based on the two main requirements: 1) System stability, and 2) Independence of the closed-loop system from changes in the constants of the components, i.e. high open-loop gain. But in the present example an additional requirement must be met: the system must have a frequency spread amenable to the Wass, Hayman method of transient response calculation (Appendix II). In fact, it was at this point in the analysis that the writer first became aware of the frequency spread requirement, by having chosen a compensator that did not satisfy it. To demonstrate: This was a lead-lag network with the transfer function:

$$C(s) = \frac{\left(1 + \frac{5}{\omega_{n_1}} s\right)\left(1 + \frac{1.25}{\omega_{n_1}} s\right)}{\left(1 + \frac{50}{\omega_{n_1}} s\right)\left(1 + \frac{.125}{\omega_{n_1}} s\right)}$$

Using the logmodang plot, Nichols chart technique (see Bibliography), the open-loop gain $K_1\ K_2$ was set at 17 db, giving a closed-loop peak gain in the upper range of acceptable values - i.e. a lightly damped transient. K_3 was set at 1.14 db to make the closed-loop steady-state gain one. The closed-loop frequency response curves are plotted in Fig. 15. The Wass, Hayman method was applied to this linear system to find the transient response for a unit step input of α_D. The fundamental frequency of the input square wave was taken as $.3\ \omega_{n_1}$, one fifth of the frequency at the resonance

peak. The 13th harmonic was included. The transient response thus derived is plotted in Fig. 16. It appears to settle down to a steady-state value of approximately .8. But the actual steady-state value is one. The method has failed, because the frequency spread is too large. We see from Fig. 15 that the frequency spread, as defined in Appendix II, is approximately:

$$\frac{6}{.08} = 75$$

Many other compensating networks were tried in an effort to obtain a satisfactory compromise between the three stated requirements. In fact, none was found, and it appears to the writer that the characteristics of the uncompensated open-loop system under consideration make the existence of such a compensator impossible.

In order to continue the investigation of the non-linear problem, the actuator, vane and aerodynamic transfer functions were arbitrarily changed to make a satisfactory compromise possible. The actuator and vane transfer functions were set equal to one, and the aerodynamic damping ratio ζ_1 was changed from .0685 to .3. The block diagram of the simplified system is shown in Fig. 17. The compensator decided upon was a lead network-amplifier with transfer function:

$$C(s) = \frac{1 + \frac{.25}{\omega_{n_1}} s}{1 + \frac{.05}{\omega_{n_1}} s}$$

$K_1 K_2$ was set at 10 db, again giving a fairly high value of closed-loop peak gain, and requiring that $K_3 = 2.38$ db for unity closed-loop steady-state gain. The closed-loop frequency response is plotted in Fig. 18. It has a satisfactory frequency spread.

SECTION 6: THE TRANSIENT RESPONSE OF THE NON-LINEAR CLOSED-LOOP SYSTEM

In the simplified system, that is for

$$W(s) = V(s) = 1$$

Eqs. 5-1 and 5-7 become, respectively,

$$E(t) = K_3 \alpha_D(t) - \alpha(t) \qquad (6\text{-}1)$$

and

$$\eta(t) = K_2 \, C(D) \, E(t) \qquad (6\text{-}2)$$

Equations 6-1 and 6-2 are applicable to the non-linear system as well as to the linear one.

Using the same notation as in Section 2, the differential equation for the aerodynamics is:

$$T(D) \alpha = U(D) \eta - \beta \alpha^3 \qquad (6\text{-}3)$$

Where

$$T(D) = D^2 + 2 \zeta_1 \omega_{n_1} D + \omega_{n_1}^2 \qquad (6\text{-}4)$$

$$U(D) = K_1 \omega_{n_1}^2 (1 + \tau D) \qquad (6\text{-}5)$$

Substituting (6-1) in (6-2) gives

$$\eta(t) = K_2 \, C(D) \left[K_3 \alpha_D(t) - \alpha(t) \right] \qquad (6\text{-}6)$$

Substituting (6-6) in (6-3),

$$T(D) \alpha = K_2 \, C(D) \, U(D) \left[K_3 \alpha_D - \alpha \right] - \beta \alpha^3 \qquad (6\text{-}7)$$

This may be rearranged in the form:

$$\left[T(D) + K_2\ C(D)\ U(D)\right]\alpha = K_2\ K_3\ C(D)\ U(D)\ \alpha_D - \beta\,\alpha^3 \tag{6-8}$$

Now an approximation to the solution of Eq. 6-8 is the solution $\alpha_1(t)$ of:

$$\left[T(D) + K_2\ C(D)\ U(D)\right]\alpha_1 = K_2\ K_3\ C(D)\ U(D)\alpha_D \tag{6-9}$$

In transfer function form, Eq. 6-9 is

$$\frac{\bar{\alpha}_1}{\bar{\alpha}_D}(s) = \frac{K_2\ K_3\ C(s)\ U(s)}{T(s) + K_2\ C(s)\ U(s)}$$

$$= \frac{K_2\ K_3\ C(s)\ \frac{U(s)}{T(s)}}{1 + K_2\ C(s)\ \frac{U(s)}{T(s)}} \tag{6-10}$$

Comparing Eqs. 6-4 and 6-5 with Eq. 5-6 we see that

$$\frac{U(s)}{T(s)} = K_1\ P(s) \tag{6-11}$$

Therefore:

$$\frac{\bar{\alpha}_1}{\bar{\alpha}_D}(s) = \frac{K_1\ K_2\ K_3\ C(s)\ P(s)}{1 + K_1\ K_2\ C(s)\ P(s)} \tag{6-12}$$

The right hand side of (6-12) is simply the closed-loop transfer function of the linear system (cf. Eq. 5-9), so that $\alpha_1(t)$, the first approximation to the transient response of the non-linear system, is the response of the linear system - not an unexpected result.

Now by analogy with the argument presented in Section 2, we take the second approximation to be:

$$\alpha_2(t) = R\,\alpha_1(t) \tag{6-13}$$

where

$$R = \frac{(\alpha)_{SN}}{(\alpha)_{SL}} \tag{6-14}$$

$(\alpha)_{SN}$ is the steady-state output of the non-linear system,

$(\alpha)_{SL}$ is the steady-state output of the linear system (equal to the magnitude of the input in the present example).

The third approximation to $\alpha(t)$ is $\alpha_3(t)$, the solution of:

$$\left[T(D) + K_2\, C(D)\, U(D)\right]\alpha_3 = K_2\, K_3\, C(D)\, U(D)\,\alpha_D - \beta\,\alpha_2^{\,3} \tag{6-15}$$

Or, if $\alpha_{3*}(t)$ is the solution of:

$$\left[T(D) + K_2\, C(D)\, U(D)\right]\alpha_{3*} = -\beta\,\alpha_2^{\,3} \tag{6-16}$$

then by superposition of the solutions of (6-9) and (6-16),

$$\alpha_3 = \alpha_1 + \alpha_{3*} \tag{6-17}$$

The solution of Eq. 6-16 follows the same lines as that of Eq. 2-16: we have $\alpha_1(t)$ in the form of a Fourier series:

$$\alpha_1(t) = |\alpha_D| \left[\frac{A_o}{2} + \frac{2}{\pi}\sum \frac{A_n}{n}\sin\left(n\,\omega_F t + B_n\right)\right]$$

where $|\alpha_D|$ is the magnitude of the step input, and the A's

and B's represent the frequency response of the closed-loop system. As before we let

$$\alpha_{1_U}(t) = \frac{A_o}{2} + \frac{2}{\pi}\sum \frac{A_n}{n} \sin(n\omega_F t + B_n)$$

so that

$$\alpha_1(t) = |\alpha_D| \alpha_{1_U}(t)$$

From Eq. 6-13:

$$\alpha_2(t) = R\alpha_1(t) = R|\alpha_D| \alpha_{1_U}(t)$$

and

$$\alpha_2^3 = R^3 |\alpha_D|^3 \alpha_1^3 \tag{6-18}$$

We find the Fourier series for $\alpha_{1_U}^3$ by the method of Section 2, and then, by Eq. 6-18, the series for α_2^3. By Eq. 6-16, α_{3*} is found by applying the frequency response

$$H(i\omega) \equiv \frac{\beta}{T(i\omega) + K_2 C(i\omega) U(i\omega)} \tag{6-19}$$

to the series for $-\alpha_2^3$. $H(i\omega)$ may be rearranged, using Eq. 6-11, to simplify its calculation. It becomes:

$$H(i\omega) = \frac{\beta}{K_2 K_3 C(i\omega)\ U(i\omega)} \left[\frac{K_1 K_2 K_3 C(i\omega) P(i\omega)}{1 + K_1 K_2 C(i\omega) P(i\omega)} \right] \tag{6-20}$$

The bracketed term is the linear system closed-loop frequency response.

The next approximation to $\alpha(t)$, if it is desired to go

further, is $\alpha_4(t)$, the solution of:

$$\left[T(D) + K_2\, C(D)\, U(D)\right] \alpha_4 = K_2\, K_3\, C(D)\, U(D) \alpha_D - \beta \alpha_3^{\,3} \tag{6-21}$$

Adding and subtracting $\beta \alpha_2^{\,3}$ on the right hand side of Eq. 6-21:

$$\left[T(D) + K_2\, C(D)\, U(D)\right] \alpha_4 = K_2\, K_3\, C(D)\, U(D) \alpha_D$$

$$- \beta \alpha_2^{\,3} + \beta \left(\alpha_2^{\,3} - \alpha_3^{\,3}\right) \tag{6-22}$$

Let $\alpha_{4*}(t)$ be the solution of:

$$\left[T(D) + K_2\, C(D)\, U(D)\right] \alpha_{4*} = \beta \left(\alpha_2^{\,3} - \alpha_3^{\,3}\right) \tag{6-23}$$

Then by superposition of the solutions of (6-15) and (6-23),

$$\alpha_4 = \alpha_3 + \alpha_{4*} \tag{6-24}$$

To solve Eq. 6-23, the Fourier series for $(\alpha_2^{\,3} - \alpha_3^{\,3})$ is found by numerical harmonic analysis in exactly the same way as outlined in Section 2 and demonstrated in Section 3. The frequency response $H(i\omega)$ is applied to it to give α_{4*}, and α_4 is found by adding the evaluations of α_3 and α_{4*}. Further iterations follow the same lines.

Having calculated $\alpha_1(t)$ and $\alpha_{3*}(t)$ for a step function of any magnitude, it is easy to find their values for any other input magnitude. Let the subscript "a" denote the case which has been calculated and "b" the case which is to be calculated. The relationship between the linear solutions is, of course,

$$\alpha_{1_b}(t) = \frac{|\alpha_{D_b}|}{|\alpha_{D_a}|} \alpha_{1_a}(t) \tag{6-25}$$

From Eq. 6-13,

$$\alpha_{2_a}(t) = R_a \, \alpha_{1_a}(t) \tag{6-26}$$

and

$$\alpha_{2_b}(t) = R_b \, \alpha_{1_b}(t) \tag{6-27}$$

Substituting (6-25) in (6-27):

$$\alpha_{2_b}(t) = R_b \frac{|\alpha_{D_b}|}{|\alpha_{D_a}|} \, \alpha_{1_a}(t) \tag{6-28}$$

Eliminating $\alpha_{1_a}(t)$ between (6-26) and (6-28):

$$\alpha_{2_b}(t) = \frac{R_b \, |\alpha_{D_b}|}{R_a \, |\alpha_{D_a}|} \, \alpha_{2_a}(t) \tag{6-29}$$

Thus

$$\alpha_{2_b}^3 = \left(\frac{R_b \, |\alpha_{D_b}|}{R_a \, |\alpha_{D_a}|} \right)^3 \alpha_{2_a}^3 \tag{6-30}$$

Due to the linearity of Eq. 6-16,

$$\alpha_{3*_b} = \left(\frac{R_b \, |\alpha_{D_b}|}{R_a \, |\alpha_{D_a}|} \right)^3 \alpha_{3*_a} \tag{6-31}$$

Having found $\alpha_{1_b}(t)$ and $\alpha_{3*_b}(t)$, $\alpha_{3_b}(t)$ follows from Eq. 6-17. No new Fourier evaluations have been performed.

The only simplification possible in the next iteration is that the evaluation of $\alpha_{2_b}^3$ follows from Eq. 6-13. α_{3b}^3 must be found by cubing α_{3_b} point by point, and a new harmonic analysis must be performed.

We now apply this procedure to the simplified control system, first for a step input α_D of magnitude .3 radian. The fundamental frequency ω_F is taken as $.38\,\omega_{n_1}$, one fifth of the closed-loop frequency response peak in Fig. 18. The tabulations of Fourier series and frequency responses for this example will be found in Table XIII, and the evaluations in Table IXV. The Fourier series for a square wave, magnitude .3, is first tabulated, then the closed-loop frequency response. The Fourier series for the output of the linear system, $\alpha_1(t)$, is found by applying the closed-loop frequency response to the square wave. We find the steady-state output of the non-linear system from Eq. 5-11, which, upon substitution of the constants, becomes:

$$(\alpha)_{SN} + .56\,(\alpha)^3_{SN} = |\alpha_D| \qquad (6\text{-}32)$$

For

$$(\alpha)_{SL} = |\alpha_D| = .3$$

find

$$(\alpha)_{SN} = .287$$

Thus, from Eq. 6-14,

$$R = \frac{.287}{.3} = .957$$

$\alpha_2(t)$, found by Eq. 6-13, is evaluated.

The Fourier series for $\alpha^3_{1_U}$, found by the method developed in Section 2, is tabulated.

$$R^3\,|\alpha_D|^3 = (.957)^3\,(.3)^3 = .02365$$

By Eq. 6-18, $-\alpha_2^3$ is found from $\alpha^3_{1_U}$ and tabulated. The frequency response $H(i\omega)$ is tabulated, and the Fourier series for α_{3*} is obtained by Eq. 6-16, tabulated, and evaluated. By Eq. 6-17, the evaluation for $\alpha_3(t)$ is found by adding those

for α_1 and α_{3*}.

$\alpha_2(t)$ and $\alpha_3(t)$ are plotted in Fig. 19. We note that $\alpha_3(t)$ is not very different from $\alpha_2(t)$ and is therefore probably a close approximation to $\alpha(t)$. The effect of the non-linearity is small, causing a very slight increase in frequency.

To illustrate a case where the effect of the non-linearity is large, we take $|\alpha_D| = 1$ radian. Although this is an unrealistic value of angle-of-attack, the result will be equivalent to that of a missile at low angle-of-attack but with a larger non-linearity. It is simply convenient to calculate it in this manner since the equations have been formulated on this basis. The evaluations are given in Table LXV. From Eq. 6-32 find

$$(\alpha)_{SN} = .757$$

Using the subscript "b" to denote this case, we have, from Eq. 6-14,

$$R_b = \frac{.757}{1} = .757$$

The subscript "a" denotes the .3 input case, already evaluated.

$$\frac{R_b \, |\alpha_{D_b}|}{R_a \, |\alpha_{D_a}|} = \frac{(.757)\,(1)}{(.957)\,(.3)} = 2.63$$

The evaluation of $\alpha_{2_b}(t)$, found according to Eq. 6-29, is tabulated.

$$\left(\frac{R_b \, |\alpha_{D_b}|}{R_a \, |\alpha_{D_a}|}\right)^3 = (2.63)^3 = 18.19$$

$\alpha_{3*_b}(t)$ is found by Eq. 6-31, and α_{3_b} by Eq. 6-17. α_{2_b} and α_{3_b} are plotted in Fig. 20.

One additional graph is presented in Fig. 21. Here the third approximations corresponding to both the .3 and the 1.0 input have been plotted, as has that for the .65 case, the calculations for which are not shown. These solutions have been "normalised" by dividing each ordinate by the corresponding steady-state value $(\alpha)_{SN}$. $\alpha_2(t)/(\alpha)_{SN}$, which, of course, is the same for all inputs, has also been plotted. This is a useful method for visualising the effect of increasing input magnitude on $\alpha_3(t)$. It would also be useful for plotting other approximations, or the exact solutions.

Improved approximations will not be demonstrated here, and the determination of the number of approximations necessary for acceptable convergence remains for the worker who thinks this method might be useful.

Finally, it is interesting to note that a linear closed-loop system with two inputs can be described which is equivalent to the non-linear system. The block diagram of this system is shown in Fig. 22. Comparison with Eq. 6-8 will prove its equivalence. Fig. 22 is, in fact, the general outline of a possible analog computer set-up where α^3 is found by putting α through a cubing device.

DISCUSSION

It has been mentioned that a quick method for determining the effect of non-linear stiffness was desired. Specifically, it was hoped that few terms would be required in the first iteration, and that one iteration would be sufficient for acceptable accuracy, at least in the first overshoot region. In fact, the number of significant terms was not small, and a single iteration was not sufficient no matter how many terms were retained, except for small inputs.

The frequency spread requirement, discussed in Appendix II, places a limitation on the range of systems to which the Wass, Hayman method may be applied. And the Wass, Hayman method must be applicable for the present method to work. If the frequency spread is of the order of 20, the number of terms in the Fourier series could be increased to handle it, but if the original control system designed here is at all typical, the frequency spread is likely to be at least 50, requiring at least 25 terms in the Fourier series for the linear response, and more for the non-linear response. The need would have to be great to justify the effort of evaluating such series.

In Section 5 an unrealistic value of aerodynamic damping was arbitrarily chosen to produce a control system to which the present method could be applied. It is possible that a more justifiable change in the control system could have been made, but this was not done due to a necessity to find an acceptable system quickly. It remains to be seen whether the present method can be used in any practical cases.

Although only "hard" stiffness has been considered, i.e. positive β , the method is equally applicable to soft stiffness, or negative β . In the latter case the presence of static instability at high angle-of-attack must be considered.

CONCLUSIONS

1. Use of the iterative technique developed is capable of giving the transient response, to an acceptable degree of accuracy, if the frequency response of the system meets the frequency spread requirement.

2. In an example, acceptable accuracy was obtained after two iterations. The number of iterations necessary when the non-linear effect is large was not determined.

3. The labor required to evaluate the Fourier series and to perform the numerical harmonic analyses necessary in the second and subsequent iterations is considerable, thus limiting the practical value of the method.

REFERENCES

Ref. 1: Brown-Edwards, E. G.: The Effectiveness of an All-Moving Control at the Rear of a Wing-Body-Control Configuration and an Example of Non-Linear Stiffness in the Response of a Missile, College of Aeronautics Thesis, May 1959, The College of Aeronautics, Cranfield.

Ref. 2: Wass, C. A. A., and E. G. Hayman: An Approximate Method of Deriving the Transient Response of a Linear System from the Frequency Response, R.A.E. Technical Note No. G.W. 148, 1951.

Ref. 3: Wylie, Jr., C. R.: Advanced Engineering Mathematics, Sec. 16.4, 1951, 1st ed.; McGraw-Hill Book Co., Inc., New York.

Ref. 4: Aeronautical Vest-Pocket Handbook, 1957, 8th ed., Pratt & Whitney Aircraft, East Hartford.

Ref. 5: Dennis, David H., and Bernard E. Cunningham: Forces and Moments on Pointed and Blunt-Nosed Bodies of Revolution at Mach Numbers from 2.75 to 5.00, NACA RM A52E22, 1952.

BIBLIOGRAPHY

Chestnut, H., and R. W. Mayer: Servomechanisms and Regulating System Design, Volume 1, 1951, John Wiley & Sons, Inc., New York.

Brown, Gordon S., and Donald P. Campbell: Principles of Servomechanisms, 1948, John Wiley & Sons, Inc., New York.

Spearman, F. R. J.: The Derivation and Use of Aerodynamic Transfer Functions of Airframes, Journal of the Royal Aeronautical Society, November 1955, pp 743 - 761.

The College of Aeronautics

EXPERIMENTAL RECORD

EXPERIMENT: TABULATION OF CONSTANTS NAME

REMARKS: TABLE I DATE

MISSILE			VANE		ACTUATOR	
	SEA LEVEL	50,000'		50,000'		
ω_{m_1} sec^{-1}	24.4	8.23	ω_{m_2} sec^{-1}	86.9	ω_{m_3} sec^{-1}	179
$\omega_{m_1}^2$ sec^{-2}	595	67.8	$\omega_{m_2}^2$ sec^{-2}	7530	$\omega_{m_3}^2$ sec^{-2}	32,000
ζ_1	.175	.0685	ζ_2	.5	ζ_3	.56
K_1	.857	.853				
τ sec	.0013	.0015				
β sec^{-1}	1386	158				

$i = 0$

j \ k	11	9	7	5	3	1	0
0	.0434	.0531	.0682	.0955	.159	.478	.125
1	.111	.135	.174	.244	.406	.608	
3	.0368	.0451	.0579	.0811	.0676		
5	.0221	.0270	.0348	.0244			
7	.0158	.0193	.0125				
9	.0123	.0075					
11	.0050						

$i = 1$

j \ k	11	9	7	5	3	1
1	.0704	.0860	.111	.155	.258	.258
3	.0469	.0574	.0377	.103	.086	
5	.0282	.0344	.0443	.031		
7	.0201	.0246	.0158			
9	.0156	.00955				
11	.0064					

$i = 3$

j \ k	11	9	7	5	3
3	.00781	.00955	.0123	.0172	.00955
5	.0094	.0115	.0147	.0103	
7	.0067	.0082	.00527		
9	.00521	.00318			
11	.00213				

$i = 5$

j \ k	11	9	7	5
5	.00281	.00344	.00442	.00206
7	.00402	.00491	.00316	
9	.00312	.00191		
11	.00128			

$i = 7$

j \ k	11	9	7
7	.00144	.00176	.00075
9	.00224	.00137	
11	.000915		

$i = 9$

j \ k	11	9
9	.00087	.000354
11	.00071	

$i = 11$

j \ k	11
11	.000199

TABULATION OF THE CONSTANTS a_{ijk}

TABLE II

EXPERIMENTAL RECORD

EXPERIMENTAL: TRIG. IDENTITIES COEFFICIENTS NAME

REMARKS: TABLE III (i) DATE

i,j,k	E_0	E_2	E_4	E_6	E_8	E_{10}	E_{12}	E_{14}
0,1,1	1/2	1/2						
0,1,3		−1/2	1/2					
0,1,5			−1/2	1/2				
0,1,7				−1/2	1/2			
0,1,9					−1/2	1/2		
0,1,11						−1/2	1/2	
0,3,3	1/2			1/2				
0,3,5		−1/2			1/2			
0,3,7			−1/2			1/2		
0,3,9				−1/2			1/2	
0,5,5	1/2					1/2		
0,5,7		−1/2					1/2	
0,5,9			−1/2					1/2

EXPERIMENTAL RECORD

EXPERIMENT: TRIG. IDENTITIES - COEFFICIENTS NAME

REMARKS: TABLE IV (ii) DATE

i, j, k	E_1	E_3	E_5	E_7	E_9	E_{11}	E_{13}	E_{15}	E_{17}
1, 1, 1	3/4	-1/4							
1, 1, 3	-1/4	1/2	-1/4						
1, 1, 5		-1/4	1/2	-1/4					
1, 1, 7			-1/4	1/2	-1/4				
1, 1, 9				-1/4	1/2	-1/4			
1, 3, 3	1/2		1/4	-1/4					
1, 3, 5	-1/4	1/4		1/4	-1/4				
1, 3, 7		-1/4	1/4		1/4	-1/4			
1, 5, 5	1/2				1/4	-1/4			
1, 5, 7	-1/4	1/4				1/4	-1/4		
3, 3, 3		3/4			-1/4				
3, 3, 5	1/4		1/2			-1/4			
3, 3, 7	-1/4			1/2			-1/4		
3, 5, 5		1/2		1/4			-1/4		
3, 5, 7	1/4		1/4		1/4			-1/4	
5, 5, 5			3/4					-1/4	
5, 5, 7		1/4		1/2					-1/4

i,j,k	F_1	F_3	F_5	F_7	F_9	F_{11}	F_{13}	F_{15}	F_{17}
1,1,1	B_1	$3B_1$							
1,1,3	$-2B_1+B_3$	B_3	$2B_1+B_3$						
1,1,5		$-2B_1+B_5$	B_5	$2B_1+B_5$					
1,1,7			$-2B_1+B_7$	B_7	$2B_1+B_7$				
1,1,9				$-2B_1+B_9$	B_9	$2B_1+B_9$			
1,3,3	B_1		$-B_1+2B_3$	B_1+2B_3					
1,3,5	$-B_1-B_3+B_5$	$B_1-B_3+B_5$		$-B_1+B_3+B_5$	$B_1+B_3+B_5$				
1,3,7		$-B_1-B_3+B_7$	$B_1-B_3+B_7$		$-B_1+B_3+B_7$	$B_1+B_3+B_7$			
1,5,5	B_1				$-B_1+2B_5$	B_1+2B_5			
1,5,7	$-B_1-B_5+B_7$	$B_1-B_5+B_7$				$-B_1+B_5+B_7$	$B_1+B_5+B_7$		
3,3,3		B_3			$3B_3$				
3,3,5	$2B_3-B_5$		B_5			$2B_3+B_5$			
3,3,7	$-2B_3+B_7$			B_7			$2B_3+B_7$		
3,5,5		B_3		$-B_3+2B_5$			B_3+2B_5		
3,5,7	$B_3+B_5-B_7$		$B_3-B_5+B_7$		$-B_3+B_5+B_7$			$B_3+B_5+B_7$	
5,5,5			B_5					$3B_5$	
5,5,7		$2B_5-B_7$		B_7					$2B_5+B$

TRIG. IDENTITIES - ANGLES

TABLE III (iii)

i,j,k	F_2	F_4	F_6	F_8	F_{10}	F_{12}	F_{14}
0,1,1	$2B_1 - 90°$						
0,1,3	$-B_1 + B_3 - 90°$	$B_1 + B_3 - 90°$					
0,1,5		$-B_1 + B_5 - 90°$	$B_1 + B_5 - 90°$				
0,1,7			$-B_1 + B_7 - 90°$	$B_1 + B_7 - 90°$			
0,1,9				$-B_1 + B_9 - 90°$	$B_1 + B_9 - 90°$		
0,1,11					$-B_1 + B_{11} - 90°$	$B_1 + B_{11} - 90°$	
0,3,3			$2B_3 - 90°$				
0,3,5	$-B_3 + B_5 - 90°$			$B_3 + B_5 - 90°$			
0,3,7		$-B_3 + B_7 - 90°$			$B_3 + B_7 - 90°$		
0,3,9			$-B_3 + B_9 - 90°$			$B_3 + B_9 - 90°$	
0,5,5					$2B_5 - 90°$		
0,5,7	$-B_5 + B_7 - 90°$					$B_5 + B_7 - 90°$	
0,5,9		$-B_5 + B_9 - 90°$					$B_5 + B_9 - 90$

TRIG. IDENTITIES - ANGLES

TABLE II (iv)

A_0	
A_1	
A_3	
A_5	
A_7	
A_9	

B_1	
B_3	
B_5	
B_7	
B_9	

ijk	$A_iA_jA_k$
000	
001	
003	
005	
007	
009	
011	
013	
015	
017	
019	

ijk	$A_iA_jA_k$
033	
035	
037	
039	
055	
057	
059	
111	
113	
115	
117	
119	
133	
135	
137	
155	
157	
159	
333	
335	
337	
355	
357	
555	
557	

$n = 0$

ijk	c	cAAA
000	.125	
011	.304	
033	.0338	
055	.0122	

n	b_n	C_n
0		
1		
2		
3		
4		
5		
6		
7		
8		
9		

FORM FOR COMPUTING α^3_{iv}

TABLE IV (i)

$n = 1$

ijk	c	cAAA	F	
113	.0645		$-2B_1+B_3+180°$	
001	.478		B_1	
111	.193		B_1	
135	.0258		$-B_1-B_3+B_5+180$	
155	.0155		B_1	
133	.043		B_1	
157	.0111		$-B_1-B_5+B_7+180°$	
335	.0043		$2B_3-B_5$	
357	.0037		$B_3+B_5-B_7$	

$n = 2$

ijk	c	cAAA	F	
011	.304		$+2B_1-90°$	
013	.203		$-B_1+B_3+90°$	
035	.0406		$-B_3+B_5+90°$	
057	.0174		$-B_5+B_7+90°$	

$n = 3$

ijk	c	cAAA	F	
113	.129		B_3	
003	.159		B_3	
111	.0645		$3B_1+180°$	
135	.0258		$B_1-B_3+B_5$	
115	.0388		$-2B_1+B_5+180°$	
355	.00515		B_3	
157	.0111		$B_1-B_5+B_7$	
137	.0645		$-B_1-B_3+B_7+180°$	
333	.00716		B_3	

$n = 4$				
ijk	r	r_{AAA}	F	
013	.203		$B_1 + B_3 - 90°$	
015	.122		$-B_1 + B_5 + 90°$	
037	.0289		$-B_3 + B_7 + 90°$	
059	.0135		$-B_5 + B_9 + 90°$	

$n = 5$				
ijk	r	r_{AAA}	F	
113	.0645		$2B_1 + B_3 + 180°$	
005	.0955		B_5	
115	.0995		B_5	
133	.0215		$-B_1 + 2B_3$	
117	.0278		$-2B_1 + B_7 + 180°$	
335	.0096		B_5	
137	.0184		$B_1 - B_3 + B_7$	
357	.0037		$B_3 - B_5 + B_7$	
555	.00154		B_5	

$n = 6$				
ijk	r	r_{AAA}	F	
015	.122		$B_1 + B_5 - 90°$	
017	.087		$-B_1 + B_7 + 90°$	
033	.0338		$2B_3 - 90°$	
039	.0225		$-B_3 + B_9 + 90°$	

$n = 7$				
ijk	r	r_{AAA}	F	
007	.0682		B_7	
135	.0258		$-B_1 + B_5 + B_3$	
115	.0388		$2B_1 + B_5 + 180°$	
133	.0215		$B_1 + 2B_3 + 180°$	
355	.0026		$-B_3 + 2B_5$	
117	.0555		B_7	
557	.0022		B_7	
337	.0061		B_7	

$n = 8$				
035	.0406		$B_3 + B_5 - 90°$	
017	.087		$B_1 + B_7 - 90°$	
019	.067		$-B_1 + B_9 + 90°$	

$n = 9$				
ijk	r	r_{AAA}	F	
135	.0258		$B_1 + B_3 + B_5 + 180$	
155	.00775		$-B_1 + 2B_5$	
117	.0278		$2B_1 + B_7 + 180°$	
137	.0845		$-B_1 + B_3 + B_7$	
009	.0531		B_9	
357	.0037		$-B_3 + B_5 + B_7$	
119	.043		B_9	

TABLE IV (ii)

EXPERIMENTAL RECORD

EXPERIMENT: FOURIER SERIES & FREQ. RESPONSES OF SECTION 2

NAME

REMARKS: TABLE V

DATE

1	2	3	4	5	6	7	8	9	10	11	12	13	14
	$U(i\omega)/T(i\omega)$		.25 SQUARE WAVE	$\alpha_1(t)$		$\alpha_{1}^{3}(t)$		$-\alpha_{2}^{3}(t)$		$\beta/T(i\omega)$		$\alpha_{3*}(t)$	
n	GAIN	PHASE SHIFT	MAG.	MAG.	PHASE ANGLE	MAG.	PHASE ANGLE	MAG.	PHASE ANGLE	GAIN	PHASE SHIFT	MAG.	PHASE ANGLE
0	.853	~~-7.8°~~	.1250	.1063		.387		-.00466		2.33		-.0109	
1	.885	-3.8°	.1592	.1410	-3.8°	.505	8.9°	.00609	-171.1°	2.42	-3.8°	.0122	-174.9°
2	1	-9.5				.148	13.5	.001785	-166.5	2.74	-9.5	.0031	-176
3	1.264	-17.1	.0530	.0670	-17.1°	.308	-17.0	.00372	163	3.46	-17.1	.0129	145.9
4	1.865	-37.9				.252	-43.5	.00304	136.5	5.1	-37.9	.0155	98.6
5	2.438	-88.1	.0318	.0775	-88.1°	.424	-92.1	.00511	87.9	6.66	-88.1	.0340	-.2
6	1.400	-136.3				.229	-160.0	.00276	20	3.83	-136.3	.0106	-116.3
7	.794	-150.4	.0228	.081	-150.4	.131	-167.0	.00158	13	2.17	-150.4	.0034	-136.6
8	.513	-160.3				.131	-154.0	.00158	-26	1.40	-160.3	.0022	-6.3
9	.368	-161	.0177	.0065	-161	.074	130.9	.00089	-49.1	1.01	-161	.0009	146.9
11	.219	-164.6	.0145	.0032	-164.6								
13	.148	-166.2	.0123	.0018	-166.2								

The College of Aeronautics

EXPERIMENTAL RECORD

EXPERIMENT: EVALUATIONS - SECTION 2　　NAME

REMARKS: TABLE VI　　DATE

SEC. t	.025	.031	.041	.051	.061	.072	.082	.092	.102	.113	.123	.133	.143	.154
$\alpha_1(t)$	.017	.045	.082	.122	.168	.213	.253	.285	.312	.331	.340	.340	.334	.322
$\alpha_2(t)$	.016	.041	.075	.112	.154	.195	.232	.262	.286	.304	.312	.312	.306	.295
$\alpha_3^*(t)$					.009		.009		0	-.009	-.017	-.032	-.044	-.057
$\alpha_3(t)$					.177		.262		.312	.322	.323	.308	.290	.265
t SEC.	.164	.174	.184	.195	.205	.225	.246	.266	.287	.307	.328	.348	.369	.390
$\alpha_1(t)$	.293	.282	.259	.237	.210	.170	.147	.144	.158	.180	.209	.235	.252	.258
$\alpha_2(t)$	.268	.258	.228	.217	.193	.156	.135	.132	.145	.165	.192	.216	.231	.236
$\alpha_3^*(t)$	-.076				-.090	-.076	-.048	-.013	.019		.049	.040	.020	-.007
$\alpha_3(t)$	.223				.120	.094	.99	.131	.177		.258	.275	.272	.251
t SEC.	.410	.430	.451	.471	.492	.512	.533	.553	.574	.594	.615			
$\alpha_1(t)$	.253	.241	.227	.211	.197	.191	.193	.198	.204	.211	.223			
$\alpha_2(t)$	.232	.221	.208	.194	.181	.175	.177	.182	.187	.194	.204			
$\alpha_3^*(t)$	-.034		-.069	-.068	-.057	-.037	-.015		.019	.029	.015			
$\alpha_3(t)$	.219		.158	.143	.140	.154	.178		.223	.240	.208			

A_0	.853
A_1	.885
A_3	1.264
A_5	2.438
A_7	.794
A_9	.368

B_1	-3.8
B_3	-17.1
B_5	-88.1
B_7	-150.4
B_9	-161

ijk	$A_iA_jA_k$
000	.617
001	.641
003	.915
005	1.762
007	.575
009	.266
011	.666
013	.951
015	1.832
017	.597
019	.2765

ijk	$A_iA_jA_k$
033	1.36
035	2.62
037	.854
039	.3955
055	5.05
057	1.644
059	.762
111	.694
113	.990
115	1.91
117	.621
119	.288
133	1.414
135	2.725
137	.889
155	5.255
157	1.71
159	.793
333	2.022
335	3.9
337	1.27
355	7.51
357	2.445
555	14.48
557	4.71

$n = 0$

ijk	c	cAAA
000	.125	.0771
011	.304	.2025
033	.0338	.0460
055	.0122	.0616

n	b_n	C_n
0	.387	
1	.505	8.9
2	.148	13.5
3	.308	-17
4	.252	-43.5
5	.424	-92.1
6	.229	-160
7	.131	-167
8	.131	-154
9	.074	-130.9

~~FORM FOR~~
COMPUTING $\alpha^3_{1,0}$

TABLE ~~IV~~ VII (i)

$n = 1$

ijk	c	cAAA	F	
113	.0645	.0639	$-2B_1+B_3+180°$	170.5
001	.478	.3063	B_1	-3.8
111	.193	.134	B_1	-3.8
135	.0258	.0704	$-B_1-B_3+B_5+180$	112.8
155	.0155	.0815	B_1	-3.8
133	.043	.0608	B_1	-3.8
157	.0111	.0190	$-B_1-B_5+B_7+180°$	121.5
335	.0043	.0168	$2B_3-B_5$	53.9
357	.0037	.0090	$B_3+B_5-B_7$	45.2

$n = 2$

ijk	c	cAAA	F	
011	.304	.2025	$+2B_1-90°$	-97.6
013	.203	.1930	$-B_1+B_3+90°$	76.7
035	.0406	.1062	$-B_3+B_5+90°$	19.0
057	.0174	.0286	$-B_5+B_7+90°$	27.7

$n = 3$

ijk	c	cAAA	F	
113	.129	.1278	B_3	-17.1
003	.159	.1459	B_3	-17.1
111	.0645	.0447	$3B_1+180°$	168.6
135	.0258	.0704	$B_1-B_3+B_5$	-74.8
115	.0388	.0741	$-2B_1+B_5+180°$	99.5
355	.00515	.0387	B_3	-17.1
157	.0111	.0189	$B_1-B_5+B_7$	-66.1
137	.01845	.0164	$-B_1-B_3+B_7+180°$	50.5
333	.00716	.0145	B_3	-17.1

$n = 4$				
ijk	r	$rAAA$	F	
013	.203	.193	$B_1 + B_3 - 90°$	-110.9
015	.122	.224	$-B_1 + B_5 + 90°$	5.7
037	.0289	.025	$-B_3 + B_7 + 90°$	-43.3
059	.0135	.010	$-B_5 + B_9 + 90°$	17.1

$n = 5$				
ijk	r	$rAAA$	F	
113	.0645	.064	$2B_1 + B_3 + 180°$	155.3
005	.0955	.169	B_5	-88.1
115	.0995	.148	B_5	-88.1
133	.0215	.030	$-B_1 + 2B_3$	-30.4
117	.0278	.017	$-2B_1 + B_7 + 180°$	37.2
335	.0096	.034	B_5	-88.1
137	.0184	.016	$B_1 - B_3 + B_7$	-137.1
357	.0037	.009	$B_3 - B_5 + B_7$	-79.4
555	.00154	.022	B_5	-88.1

$n = 6$				
ijk	r	$rAAA$	F	
015	.122	.224	$B_1 + B_5 - 90°$	178.1
017	.087	.052	$-B_1 + B_7 + 90°$	-56.6
033	.0338	.046	$2B_3 - 90°$	-124.2
039	.0225	.009	$-B_3 + B_9 + 90°$	-53.9

$n = 7$				
ijk	r	$rAAA$	F	
007	.0682	.039	B_7	-150.4
135	.0258	.070	$-B_1 + B_5 + B_3$	-101.4
115	.0388	.074	$2B_1 + B_5 + 180°$	84.3
133	.0215	.030	$B_1 + 2B_3 + 180°$	143.0
355	.0026	.019	$-B_3 + 2B_5$	-159.1
117	.0555	.034	B_7	-150.4
557	.0022	.010	B_7	-150.4
337	.0061	.008	B_7	-150.4

$n = 8$				
035	.0406	.106	$B_3 + B_5 - 90°$	164.8
017	.087	.052	$B_1 + B_7 - 90°$	115.8
019	.067	.019	$-B_1 + B_9 + 90°$	-67.2

$n = 9$				
ijk	r	$rAAA$	F	
135	.0258	.070	$B_1 + B_3 + B_5 + 180$	71
155	.00775	.041	$-B_1 + 2B_5$	-172.4
117	.0278	.017	$2B_1 + B_7 + 180°$	22
137	.01845	.016	$-B_1 + B_3 + B_7$	-163.7
009	.0531	.026	B_9	-161
357	.0037	.009	$-B_3 + B_5 + B_7$	138.6
119	.043	.012	B_9	-161

TABLE ~~IV~~ VII (ii)

EXPERIMENTAL RECORD

EXPERIMENT: FOURIER SERIES & FREQ. RESP. **NAME**

REMARKS: TABLE VIII **DATE**

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
	$V(i\omega)/F(i\omega)$		$\eta(t)$	$\alpha_1(t)$		$\alpha_{1\nu}^3(t)$		$-\alpha_2^3(t)$		$\alpha_{3*}(t)$		$(\alpha_2^3-\alpha_3^3)$		$\alpha_{4*}(t)$	
n	GAIN	PHASE SHIFT	MAG.	MAG.	PHASE ANGLE	MAG.	PHASE ANGLE	MAG.	PHASE ANGLE	MAG.	PHASE ANGLE	MAG.	PHASE ANGLE	MAG.	PHASE ANGLE
0	1		.2	.2		.549		-.0246		.0246		.00035		.0003	
1	1.033	-7.1°	.2545	.263	-7.1°	.710	-1°	.0318	179	.329	171.9	.00079	25.5	.0008	18.4
2	1.173	-16				.130	5	.00581	-175	.0068	169	.00193	-17.2	.0023	-33.2
3	1.360	-29.8	.0849	.117	-29.8	.386	-27.1	.0173	152.9	.0238	123.1	.00842	-66.8	.0061	-96.6
4	1.668	-53.1				.224	-71	.0100	109°	.0167	55.9	.00578	-133	.0096	-186.1
5	1.668	-90	.0509	.0849	-90	.307	-96.9	.0137	83.1	.0228	-6.9	.00825	146	.0138	-304.0
6	1.185	-119.1				.209	-163.4	.00935	16.6	.0111	-102.5	.00564	85.8	.0067	-33.3
7	.784	-138.8	.0364	.0285	-138.8	.121	-165.7	.00536	15.1	.0042	-123.7	.00254	53.7	.0020	-85.1
8	.546	-148.4				.131	142.5	.00585	-37	.0032	-185.4	.00289	35	.0016	-113.4
9	.402	-154.3	.0283	.0114	-154.3	.044	126.8	.00197	-53.2	.0008	-207.5	.00212	11.9	.0009	-142.4
11	.246	-161	.0232	.0057	-161										
13	.1672	-164.8	.0196	.0033	-164.8										

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EXPERIMENTAL RECORD

EXPERIMENT: EVALUATIONS - SECTION 3 NAME

REMARKS: TABLE IX DATE

t	1	2	2.5	2.75	3	3.25	3.5	~~4~~ 3.75	5	5.5	6	6.5	7	8
$\alpha_1(t)$	.148	.408	.500	.527	.544	.550	.547	.540	.424	.383	.353	.347	.348	.376
$\alpha_2(t)$	.131	.362	.444	.468	.482	.489	.485	.480	.376	.340	.313	.308	.309	.334
$\alpha_3*(t)$	.002	-.003	-.018	-.029	-.042	-.058	-.074	-.091	-.134	-.122	-.096	-.062	-.029	.006
$\alpha_3(t)$	.150	.405	.482	.498	.502	.492	.473	.449	.290	.261	.257	.285	.319	.382
$\alpha_2^3(t)$	.0023	.0474	.0875	.1025	.1120	.1160	.1150	.1106	.0532	.0393	.0307	.0292	.0295	.0373
$\alpha_3^3(t)$	.0034	.0664	.1120	.1235	.1265	.1191	.1058	.0905	.0244	.0178	.0170	.0231	.0325	.0557
$\alpha_2^3-\alpha_3^3$	-.0011	-.0190	-.0245	-.0210	-.0145	-.0031	.0092	.0201	.0288	.0215	.0137	.0061	-.0030	-.0184
$\alpha_4*(t)$	.004	.002	-.005	-.009	-.012	-.013	-.017	-.018	.002	.021	.033	.038	.035	.008
$\alpha_4(t)$	.154	.407	.477	.489	.490	.479	.456	.431	.292	.282	.290	.323	.354	.390

t	9	10	11	12	13	14	15		4
$\alpha_1(t)$	.410	.416	.412	.397	.391	.393	.403		.520
$\alpha_2(t)$	.364	.370	.366	.352	.347	.349	.358		.461
$\alpha_3*(t)$	-.008	-.035	-.067	-.068	-.053	-.036	-.034		-.104
$\alpha_3(t)$	.402	.380	.345	.329	.338	.357	.369		.416
$\alpha_2^3(t)$	.0482	.0507	.0490	.0436	.0418	.0425	.0459		.0980
$\alpha_3^3(t)$	.0650	.0549	.0411	.0356	.0386	.0455	.0502		.0720
$\alpha_2^3-\alpha_3^3$	-.0168	-.0042	.0079	.0080	.0032	-.0030	-.0043		.0260
$\alpha_4*(t)$	-.022	-.033	-.019	.006	.021	.016	0		-.017
$\alpha_4(t)$	.380	.347	.326	.335	.359	.373	.369		.399

A_0	1
A_1	1.033
A_3	1.378
A_5	1.668
A_7	.784
A_9	.402

B_1	-7.1
B_3	-29.8
B_5	-90
B_7	-138.8
B_9	-154.3

ijk	$A_iA_jA_k$
000	1
001	1.033
003	1.378
005	1.668
007	.784
009	.402
011	1.07
013	1.422
015	1.721
017	.81
019	.416

ijk	$A_iA_jA_k$
033	1.896
035	2.295
037	1.079
039	.554
055	2.78
057	1.307
059	.67
111	1.105
113	1.47
115	1.78
117	.838
119	.43
133	1.958
135	2.37
137	1.112
155	2.87
157	1.35
159	.693
333	2.605
335	3.16
337	1.484
355	3.825
357	1.8
555	4.63
557	2.175

$n = 0$		
ijk	c	cAAA
000	.125	.125
011	.304	.326
033	.0338	.064
055	.0122	.034

n	b_n	C_n
0	.549	
1	.710	-1°
2	.130	5
3	.386	-27.1
4	.224	-71
5	.307	-96.9
6	.209	-163.4
7	.121	-165.7
8	.131	142.5
9	.044	126.8

~~FORM FOR~~
COMPUTING α^3_{iv}

TABLE ~~IX~~ X (i)

$n = 1$				
ijk	c	cAAA	F	
113	.0645	.0948	$-2B_1+B_3+180°$	164.4
001	.478	.494	B_1	-7.1
111	.193	.213	B_1	-7.1
135	.0258	.061	$-B_1-B_3+B_5+180$	126.9
155	.0155	.0445	B_1	-7.1
133	.043	.084	B_1	-7.1
157	.0111	.015	$-B_1-B_5+B_7+180°$	138.3
335	.0043	.014	$2B_3-B_5$	30.4
357	.0037	.007	$B_3+B_5-B_7$	19.0

$n = 2$				
ijk	c	cAAA	F	
011	.304	.326	$+2B_1-90°$	-104.2
013	.203	.288	$-B_1+B_3+90°$	67.3
035	.0406	.093	$-B_3+B_5+90°$	29.8
057	.0174	.023	$-B_5+B_7+90°$	41.2

$n = 3$				
ijk	c	cAAA	F	
113	.129	.190	B_3	-29.8
003	.159	.219	B_3	-29.8
111	.0645	.071	$3B_1+180°$	158.7
135	.0258	.061	$B_1-B_3+B_5$	-67.3
115	.0388	.069	$-2B_1+B_5+180$	104.2
355	.00515	.020	B_3	-29.8
157	.0111	.015	$B_1-B_5+B_7$	-55.9
137	.01845	.021	$-B_1-B_3+B_7+180$	78.1
333	.00716	.019	B_3	-29.8

$n = 4$				
ijk	τ	τAAA	F	
013	.203	.289	$B_1 + B_3 - 90°$	-126.9
015	.122	.210	$-B_1 + B_5 + 90°$	7.1
037	.0289	.031	$-B_3 + B_7 + 90°$	-19
059	.0135	.009	$-B_5 + B_9 + 90°$	25.7

$n = 5$				
ijk	τ	τAAA	F	
113	.0645	.095	$2B_1 + B_3 + 180°$	136
005	.0955	.159	B_5	-90
115	.0995	.138	B_5	-90
133	.0215	.042	$-B_1 + 2B_3$	-52.5
117	.0278	.023	$-2B_1 + B_7 + 180°$	55.4
335	.0096	.027	B_5	-90
137	.0184	.020	$B_1 - B_3 + B_7$	-116.1
357	.0037	.007	$B_3 - B_5 + B_7$	-78.6
555	.00154	.007	B_5	-90

$n = 6$				
ijk	τ	τAAA	F	
015	.122	.210	$B_1 + B_5 - 90°$	172.9
017	.087	.070	$-B_1 + B_7 + 90°$	-44.7
033	.0338	.064	$2B_3 - 90°$	-149.6
039	.0225	.012	$-B_3 + B_9 + 90°$	-34.5

$n = 7$				
ijk	τ	τAAA	F	
007	.0682	.053	B_7	-138.8
135	.0258	.061	$-B_1 + B_5 + B_3$	-112.7
115	.0388	.069	$2B_1 + B_5 + 180°$	75.8
133	.0215	.042	$B_1 + 2B_3 + 180°$	113.3
355	.0026	.010	$-B_3 + 2B_5$	-150.2
117	.0555	.046	B_7	-138.8
557	.0022	.005	B_7	-138.8
337	.0061	.009	B_7	-138.5

$n = 8$				
035	.0406	.093	$B_3 + B_5 - 90°$	150.2
017	.087	.070	$B_1 + B_7 - 90°$	124.1
019	.067	.028	$-B_1 + B_9 + 90°$	-57.2

$n = 9$				
ijk	τ	τAAA	F	
135	.0258	.061	$B_1 + B_3 + B_5 + 180$	53.1
155	.00775	.022	$-B_1 + 2B_5$	-172.9
117	.0278	.023	$2B_1 + B_7 + 180°$	27
137	.0845	.021	$-B_1 + B_3 + B_7$	-161.5
009	.0531	.021	B_9	-154.3
357	.0037	.007	$-B_3 + B_5 + B_7$	161
119	.043	.018	B_9	-154.3

TABLE ~~IX~~ X (ii)

EXPERIMENTAL RECORD

EXPERIMENT: EVALUATIONS - SECOND HALF-CYCLE NAME

REMARKS: TABLE XI DATE

t_1	1	2	2.5	2.75	3	3.25	3.5	3.75	4	5	5.5	6	6.5	7
$\alpha_2(t_1)$	.224	-.007	-.089	-.113	-.127	-.133	-.131	-.125	-.106	-.021	.015	.042	.047	.046
$\alpha_3(t_1)$	.272	-.032	-.111	-.132	-.143	-.143	-.136	-.126	-.104	-.011	.025	.050	.051	.048
$\alpha_2^3(t_1)$	.0112	0	-.0007	.0014	-.0020	-.0024	.0022	-.0020	-.0012	0	0	.0011	.0011	.0001
$\alpha_3^3(t_1)$	.0095	0	-.0014	-.0023	-.0029	-.0029	-.0025	-.0020	-.0011	0	0	.0001	.0001	.0001
$\alpha_2^3-\alpha_3^3$	.0017	0	.0007	.0009	.0009	.0005	.0003	0	-.0001	0	0	0	0	0

t_1	8	9	10	11	12	13	14	15
$\alpha_2(t_1)$	.021	-.009	-.015	-.011	.003	.008	.006	-.003
$\alpha_3(t_1)$	.018	-.014	-.016	-.009	.004	.009	.008	-.003
$\alpha_2^3(t_1)$	0	0	0	0	0	0	0	0
$\alpha_3^3(t_1)$	0	0	0	0	0	0	0	0
$\alpha_2^3-\alpha_3^3$	0	0	0	0	0	0	0	0

EXPERIMENTAL RECORD

EXPERIMENT: HARMONIC ANALYSIS NAME

REMARKS: TABLE XII (i) DATE

y_0 TO y_{12}	0	-55	-230	240	260	55	-180	-155	35	88	30	-46	0
y_{23} TO y_{13}	0 →												
SUMS: (c_0 - c_{12})	0	-55	-230	240	260	55	-180	-155	35	88	30	-46	0
Diffs: (d_1 - d_{11})	0	-55	-230	240	260	55	-180	-155	35	88	30	-46	0
c_0 to c_6	0	-55	-230	240	260	55	-180						
c_{12} to c_7	0	-46	30	88	35	-155							
Sums: (e_0 TO e_6)	0	-101	-200	328	295	-100	-180						
Diffs: (f_0 TO f_5)	0	-9	-260	152	225	210	—						
(d_1 TO d_6)	-55	-230	240	260	55	-180							
(d_{11} TO d_7)	-46	30	88	35	-155	—							
SUMS: (g_1 TO g_6)	-101	-200	328	295	-100	-180							
DIFFS: (h_1 TO h_5)	-9	-260	152	225	210	—							
(e_0 TO e_3)	0	-101	-200	328		(h_1 TO h_3)			-9	-260	152		
(e_6 TO e_4)	-180	-100	295			(h_5 TO h_4)			210	225			
SUMS: (j_0 TO j_3)	-180	-201	95	328		SUMS: (l_1 TO l_3)			201	-35	152		
DIFFS: (k_0 - k_2)	180	-1	-495	—		DIFFS: (m_1 - m_2)			-219	-485	—		

EXPERIMENTAL RECORD

EXPERIMENT: HARMONIC ANALYSIS NAME

REMARKS: TABLE XII (ii) DATE

	j_1 -201	j_2 95	k_2 -495	l_1 201			k_1 -1	l_2 -35	m_1 -219	m_2 -485				
× ½	j_1' -100.5	47.5	-247.5	100.5		×.866	-.866	-30.3	-189.8	-420				
	j_0 =	-180	j_1 =	-201	k_0 =	180	k_1' =	-.866	j_0 =	-180	j_1' =	-100.5	k_0 =	180
	j_2 =	95	j_3 =	328	k_2' =	-247.5			$-j_2'$ =	-47.5	j_3 =	-328	k_2 =	-495
Sum Col 1		-85				-67.5				-227.5				
Sum Col 2		127				-.9				-428.5				
Sum		42	$=12a_0$			-68.4	$=12a_2$			-656	$=12a_4$			
Diff		-212	$=24a_{12}$			-66.6	$=12a_{10}$			201	$=12a_8$		$12a_6$ =	675
	l_1' =	100.5	l_2' =	-30.3	m_1' =	-189.8		l_1 =	201					
	l_3 =	152			m_2' =	-420		l_3 =	152					
Sum Col 1		252.5												
Sum Col 2		-30.3												
Sum		222.2	$=12b_2$			-609.8	$=12b_4$							
Diff		282.8	$=12b_{10}$			230.2	$=12b_8$		49	$=12b_6$				
	f_1	-9	f_3	152	f_5	+210	g_1	-101	g_3	328	g_5	-100		
× .707		-6.4		107.5		+148.4		-71.4		232		-70.7		

EXPERIMENTAL RECORD

EXPERIMENT: HARMONIC ANALYSIS **NAME:**

REMARKS: TABLE XII (iii) **DATE:**

	f_1'	-6.4				g_1'	-71.4	
	f_5'	148.4				g_5'	-70.7	
SUM		142	$= p_1$		SUM		-142.1	$= s_1$
DIFF		-154.8	$= p_2$		DIFF		-.7	$= s_2$
	f_2	-260	p_1	142	g_4	295	s_1	-142.1
x.866		-225		123		255.5		-123
	f_4	225	p_2	-154.8	g_2	-200	s_2	-.7
x 1/2		112.5		-77.4		-100		-.3
	f_4'	112.5	p_1'	123	g_3'	232	g_6	-180
	f_2'	-225	p_2'	-77.4	s_2'	-.3	g_2'	-100
SUM	$q_1 =$	-112.5	$r_1 =$	45.6			$u =$	-280
DIFF	$q_2 =$	337.5	$r_2 =$	200.4	$t =$	232.3		

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EXPERIMENTAL RECORD

EXPERIMENT: HARMONIC ANALYSIS NAME

REMARKS: TABLE XII (iv) DATE

	f_0	0	f_3'	107.5	f_0	0	p_2	-154.8	f_0	0	r_2	200.4	
	g_1	-112.5	r_1	45.6	$-f_4$	-225	$-f_3'$	-107.5	g_2	337.5	$-f_3'$	-107.5	
SUM COL 1		-112.5				-225				337.5			
SUM COL 2		153.1				-262.3				92.9			
SUM		40.6	$=12a_1$			-487.3	$=12a_3$			430.4	$=12a_5$		
DIFF		-265.6	$=12a_{11}$			37.3	$=12a_9$			244.6	$=12a_7$		
	s_1'	-123	u	-280	s_2	-.7	g_2	-200	s_1'	-123	u	-280	
	t	232.3	g_4'	255.5	g_3'	232	$-g_6$	180	$-t$	-232.3	$-g_4'$	-255.5	
		109.3				231.3				-355.3			
		-24.5				-20				-535.5			
		84.8	$=12b_1$			211.3	$=12b_3$			-890.8	$=12b_5$		
		133.8	$=12b_{11}$			251.3	$=12b_9$			180.2	$=12b_7$		
n	0	1	2	3	4	5	6	7	8	9	10	11	12
$12a_n$	42	40.6	-69.4	-487.3	-656	430.4	675	244.6	201	37.3	-66.6	-265.6	-106
$12b_n$	~~0~~	84.8	222.2	211.3	-609.8	-890.8	49	180.2	230.2	251.3	282.8	133.8	

DIVIDE BY 1000, THEN $(\alpha_2^3 - \alpha_3^3) = \frac{1}{2}a_0 + a_1 \cos \omega_F t + a_2 \cos 2\omega_F t + \ldots$

$+ b_1 \sin \omega_F t + b_2 \sin 2\omega_F t + \ldots$

EXPERIMENTAL RECORD

EXPERIMENT: FOURIER SERIES & FREQ. RESP. NAME

REMARKS: TABLE XIII DATE

	3 SQUARE WAVE	CLOSED-LOOP FREQ. RESP.		$\alpha_1(t)$		$\alpha^3_{1U}(t)$		$-\alpha^3_2(t)$		$H(i\omega)$		$\alpha_{3*}(t)$	
n	MAG.	GAIN	PHASE SHIFT	MAG.	PHASE ANGLE	MAG.	PHASE ANGLE	MAG.	PHASE ANGLE	GAIN	PHASE SHIFT	MAG.	PHASE ANGLE
0	.1500	1		.15		.551		-.01305		.56		-.0073	
1	.1910	1.030	-2.8°	.197	-2.8°	.702	4.0°	.0166	-176°	.575	-7.3	.00955	-183.3°
2		1.130	-6.1			.135	25.9	.00320	-164.1	.619	-14.6	.00198	-168.7
3	.0636	1.344	-14.1	.0855	-14.1	.381	-11.9	.00900	+168.1	.728	-26.6	.00655	141.5
4		1.656	-28.3			.194	-36.9	.00459	143.1	.867	-44.8	.00398	98.3
5	.0382	1.890	-54.9	.0723	-54.9	.346	-59.6	.00820	120.4	.961	-74.9	.00788	45.5
6		1.638	-83.5			.232	-117.1	.00550	62.9	.803	-106.7	.00441	-43.8
7	.0273	1.17	-104	.0319	-104	.176	-119.8	.00417	60.2	.551	-130.1	.00230	-69.9
8		.839	-114			.159	-170.4	.00376	9.6	.378	-142.6	.00142	-133.0
9	.0212	.645	-118.5	.0137	-118.5	.073	-164.2	.00174	15.8	.279	-149.4	.00049	-133.6
11	.0174	.431	-123.5	.0075	-123.5								
13	.0147	.312	-125.8	.00346	-125.8								

EXPERIMENTAL RECORD

EXPERIMENT: EVALUATIONS - SECTION 6

NAME

REMARKS: TABLE IXV

DATE

t SECS	.048	.064	.112	.128	.144	.160	.175	.191	.208	.224	.239	.255	.303	.319
.3 INPUT α_1	.137	.199	.365	.398	.417	.425	.422	.414	.398	.381	.358	.338	.280	.266
α_2	.132	.191	.350	.381	.400	.407	.404	.396	.381	.364	.344	.324	.268	.254
α_3^*	.0028	.0016	-.0020	-.0053	-.0093	-.0145	-.0202	-.0257	-.0312	-.0357	-.0393	-.0413	-.0371	-.0328
α_3	.1365	.201	.363	.393	.408	.4105	.402	.388	.367	.345	.319	.297	.243	.233
1.0 INPUT α_1	.456	.662	1.214	1.324	1.390	1.413	1.406	1.343	1.324	1.269	1.191	1.125	.931	.885
α_2	.345	.501	.920	1.002	1.051	1.070	1.063	1.042	1.002	.960	.902	.851	.705	.676
α_3^*	.045	.029	-.036	-.096	-.169	.264	-.368	-.467	-.567	-.650	-.715	-.751	-.675	-.596
α_3	.501	.691	1.178	1.228	1.221	1.149	1.038	.876	.757	.619	.476	.374	.256	.289

t SECS	.351	.383	.415	.447	.510	.575	.639	.702	.765	.830	.894	.956
.3 INPUT α_1	.253	.258	.272	.286	.312	.316	.304	.297	.291	.302	.298	.312
α_2	.242	.246	.260	.274	.298	.302	.290	.284	.278	.289	.286	.300
α_3^*	-.0212	-.0096	-.0006	.0031	-.0024	-.0149	-.0207		-.0127	-.0093	-.0106	-.0143
α_3	.232	.248	.271	.289	.309	.301	.283		.278	.293	.287	.298
1.0 INPUT α_1	.842	.859	.905	.951	1.039	1.051	1.011		.969	1.005	.991	1.039
α_2	.638	.650	.685	.721	.786	.796	.766		.734	.761	.751	.786
α_3^*	-.386	-.175	-.011	.056	-.051	-.271	-.376		-.231	-.169	-.191	-.260
α_3	.456	.684	.894	1.007	.988	.780	.635		.738	.836	.800	.779

EXPERIMENTAL RECORD

EXPERIMENT: THE "φ" TABLE FOR EVEN HARMONICS NAME

REMARKS: TABLE XV DATE

t/T \ n	2	4	6	8
.50	11.5	22.9	34.4	45.8
.75	17.2	34.4	51.5	68.8
1.00	22.9	45.9	68.8	91.7
1.25	28.6	57.3	85.9	114.7
1.50	34.4	68.8	103.0	137.6
1.75	40.1	80.2	120.2	160.5
2.00	45.8	91.7	137.5	183.4
2.25	51.5	103.0	154.8	206.2
2.50	57.3	114.6	171.9	229.3
2.75	63.0	126.0	189.0	252.1
3.00	68.7	137.6	206.1	275.2
3.25	74.5	149.0	223.4	298.0
3.5	80.2	160.3	240.6	321.0
3.75	85.9	171.9	257.7	344.0
4.00	91.6	183.2	275.0	6.6
4.25	97.4	194.8	292.2	29.7
4.50	103.1	206.2	309.4	52.4
4.75	108.8	217.8	326.6	75.6
5.00	114.5	229.2	343.8	98.4
5.50	126.0	252.0	18.2	144.0
6.00	137.0	275.0	52.6	190.2
6.50	148.9	298.0	87.0	236.0

t/T \ n	2	4	6	8
7.00	160.4	321.0	121.2	281.8
7.50	171.8	343.8	155.6	327.7
8.00	183.4	6.6	190.0	13.5
8.50	194.8	29.8	224.0	59.4
9.00	206.2	52.5	258.8	105.2
9.50	217.8	75.5	293.2	151.1
10.0	229.2	98.5	327.6	196.9
10.5	240.6	121.5	2.0	242.8
11.0	252.0	144.0	36.4	288.6
11.5	263.6	167.5	70.6	334.5
12.0	275.0	190.0	105.0	20.3
12.5	286.4	213.0	139.4	66.2
13.0	298.0	236.5	173.8	112.0
13.5	309.4	259.0	208.2	157.8
14.0	320.8	282.0	242.6	203.7
14.5	332.4	305.0	277.0	249.5
15.0	343.8	328	311.4	295.4

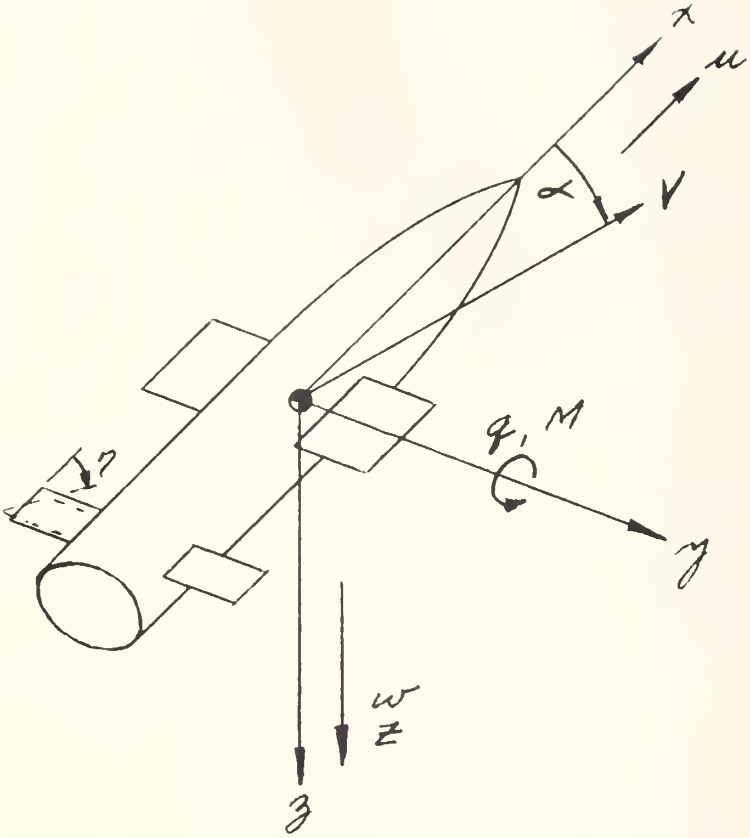


FIG. 1

AXES AND SIGN CONVENTIONS

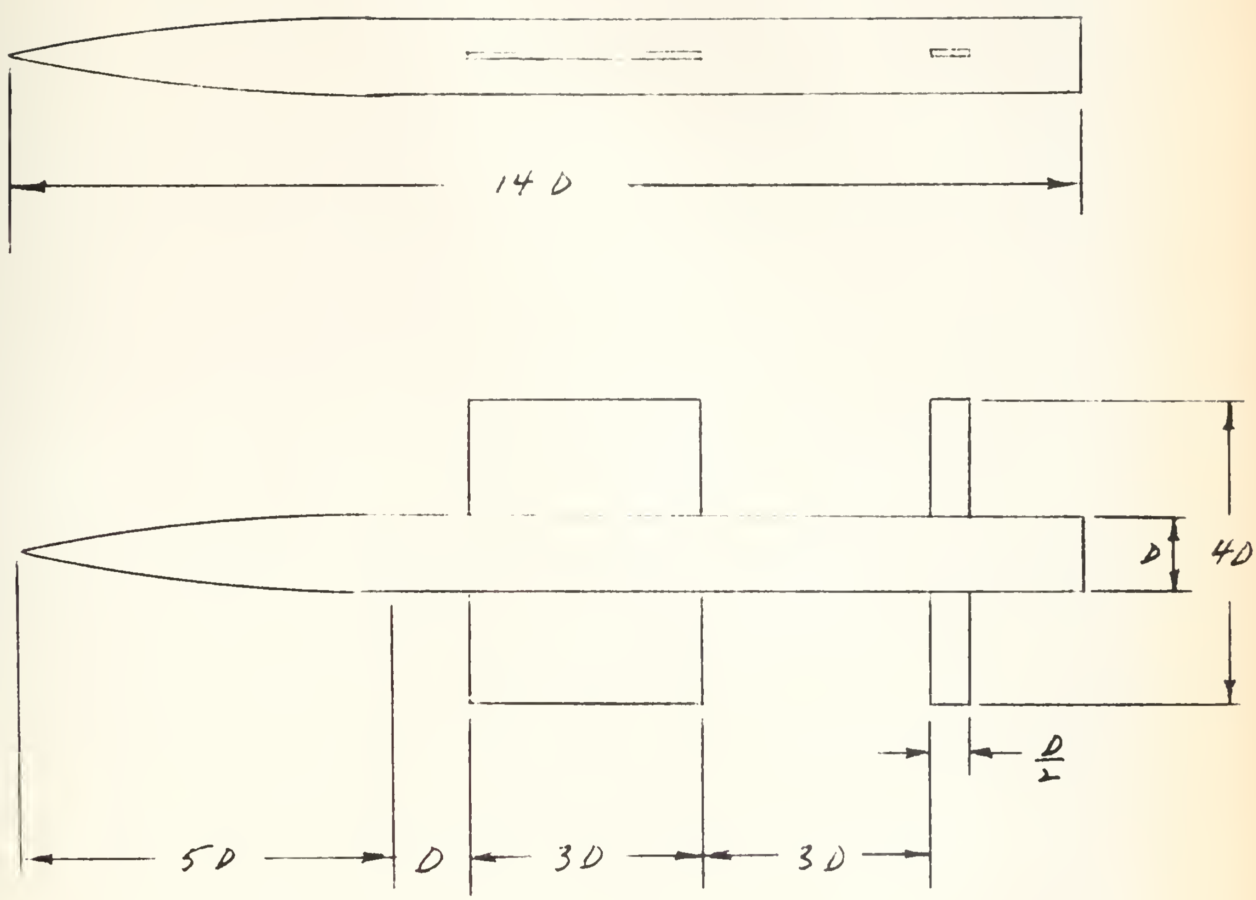


FIG. 2

MISSILE CONFIGURATION

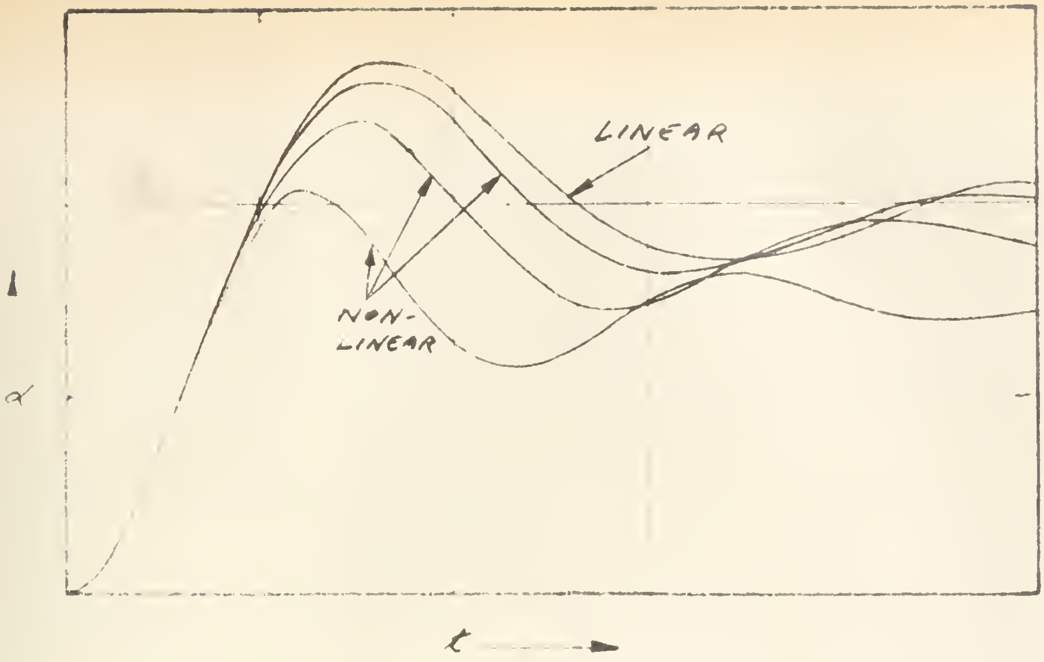
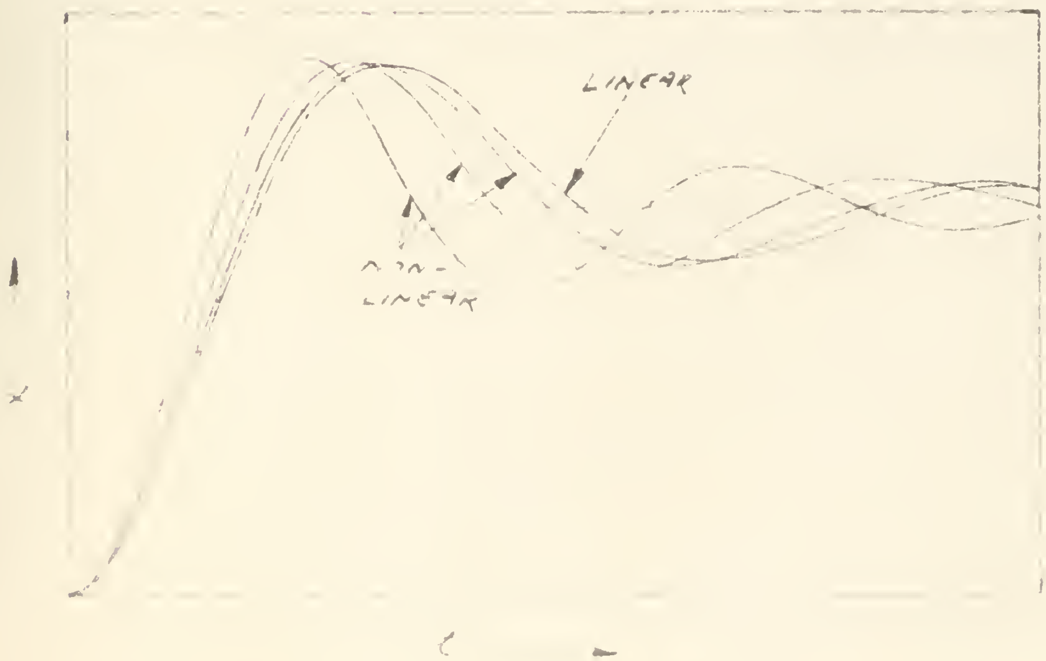


FIG. 3

COMPUTER SOLUTIONS



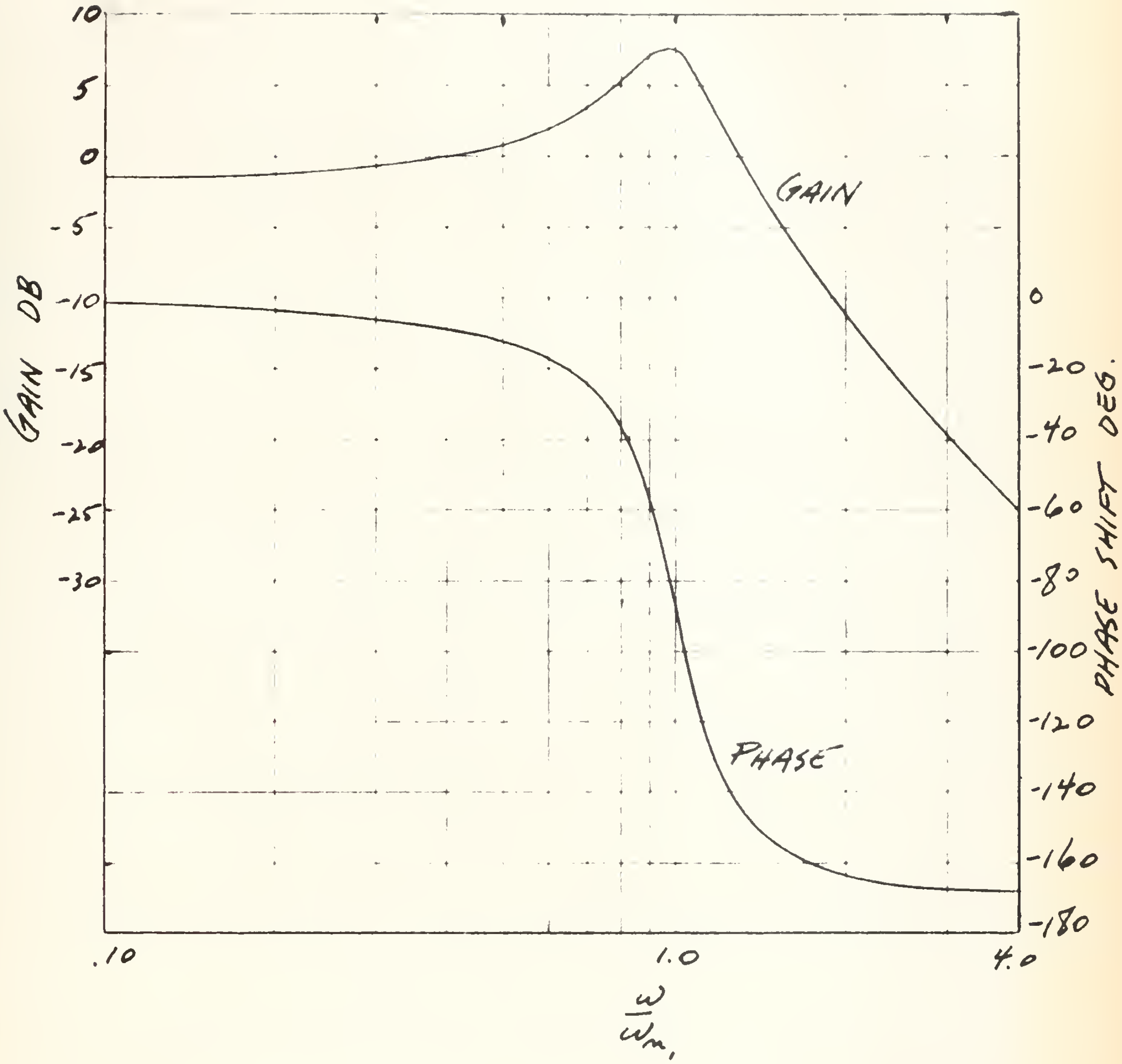
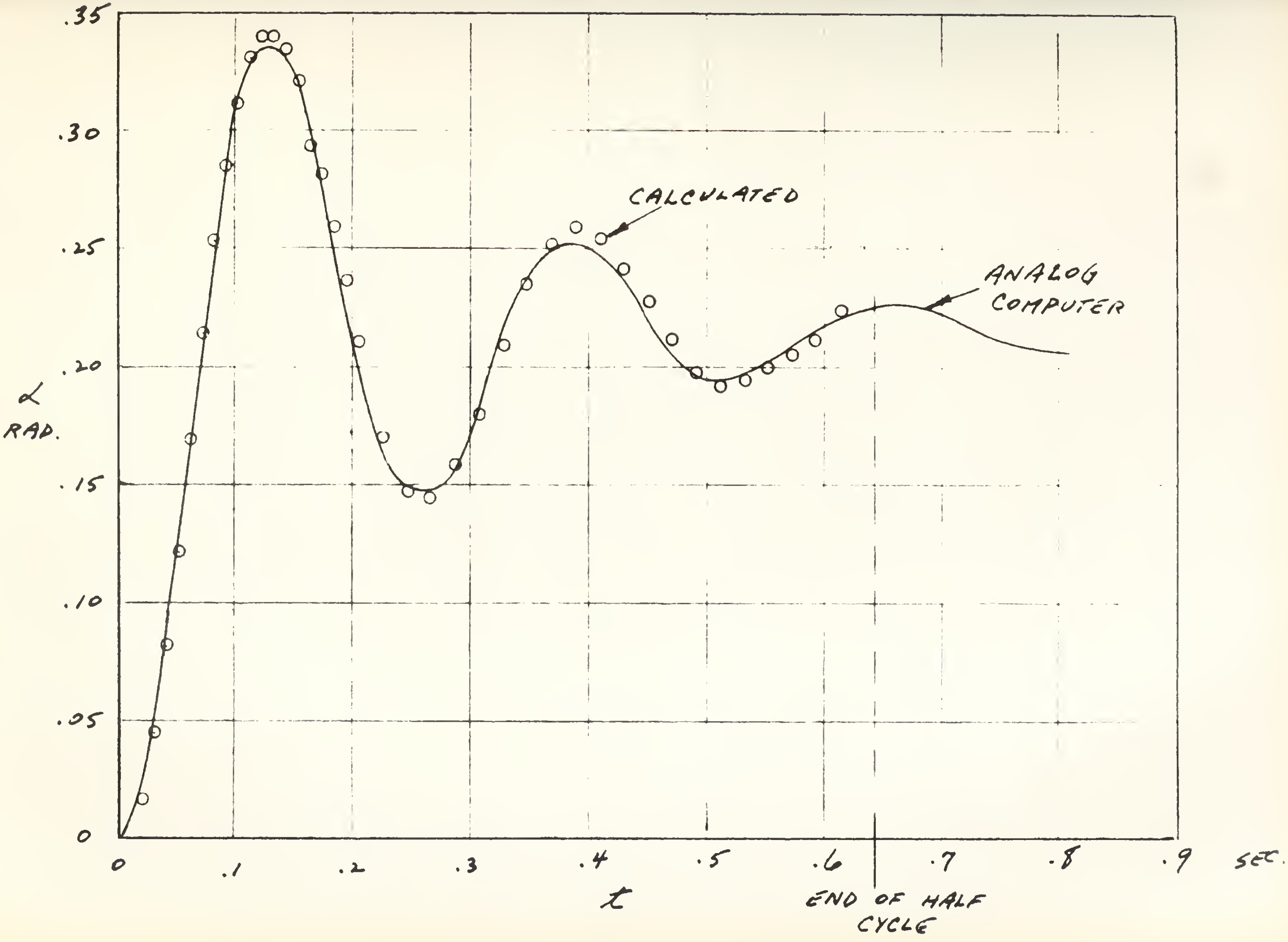


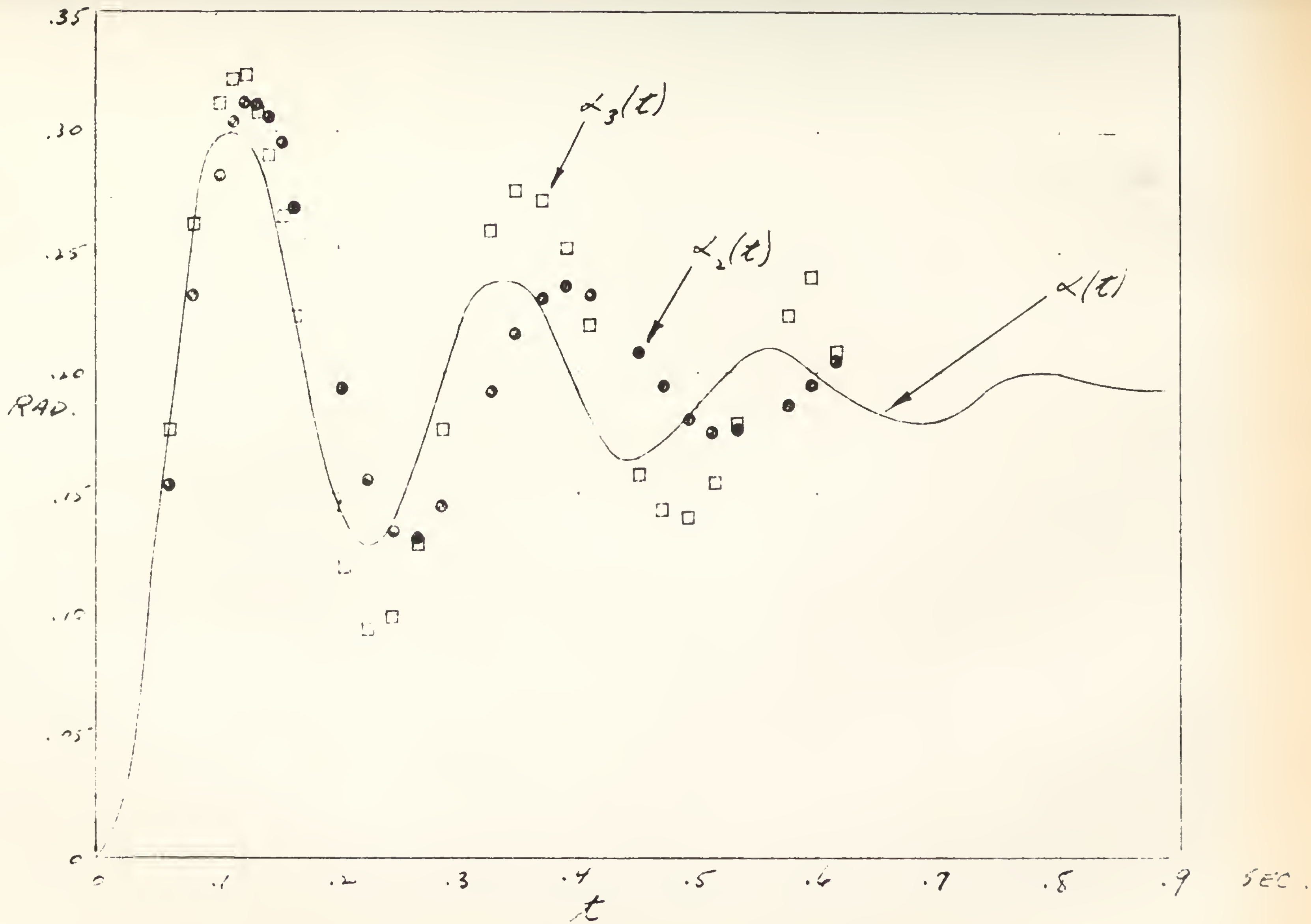
FIG. 5

FREQUENCY RESPONSE, SEA LEVEL



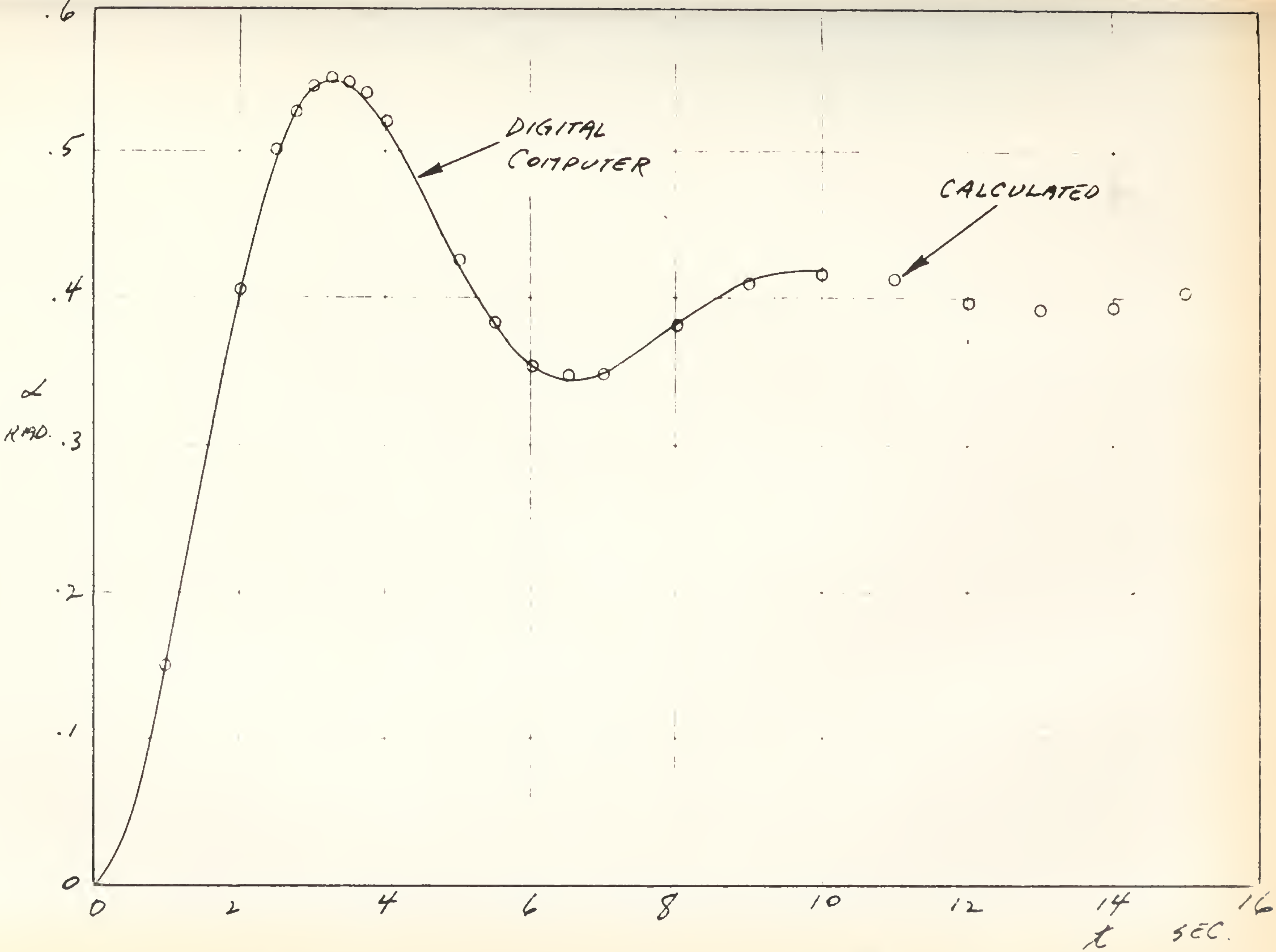
LINEAR TRANSIENT RESPONSE TO ELEVATOR STEP

FIG. 6



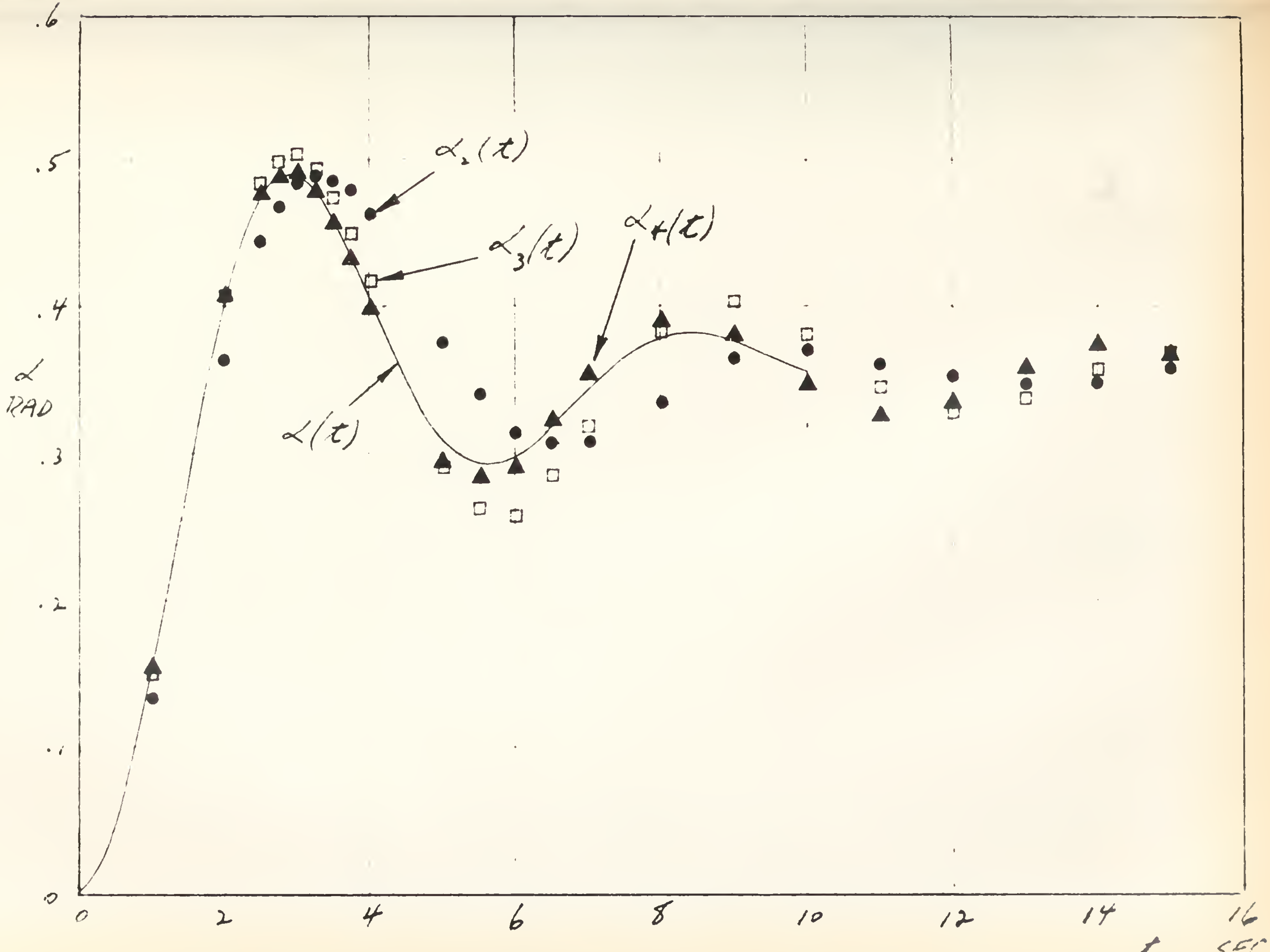
NON-LINEAR TRANSIENT RESPONSE TO ELEVATOR STEP

FIG. 7



DIGITAL COMPUTER EQUATION - LINEAR TRANSIENT

FIG. 8



DIGITAL COMPUTER EQUATION - NON LINEAR TRANSIENT

FIG. 9

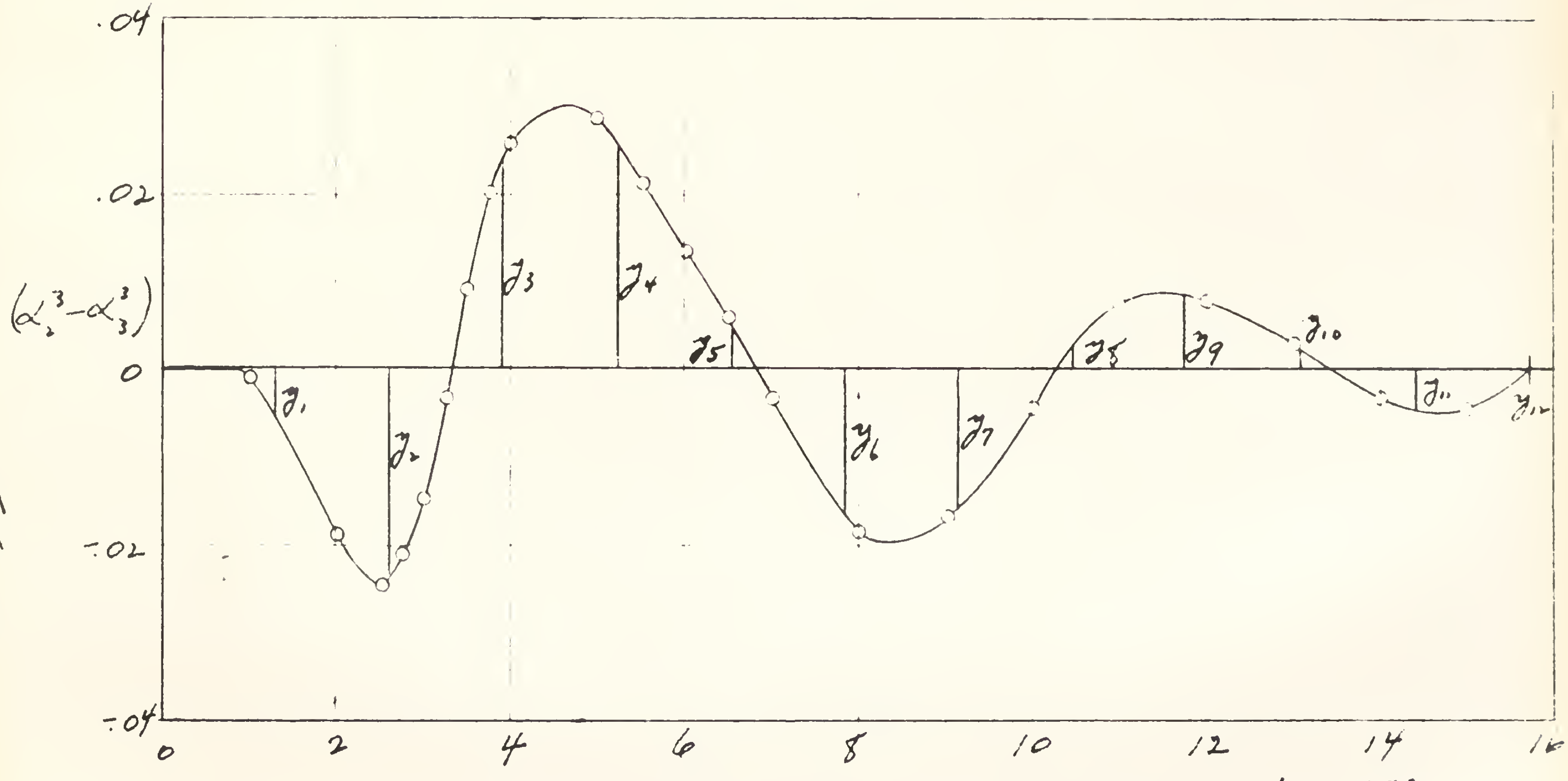


FIG. 10

DETERMINATION OF ORDINATES FOR HARMONIC ANALYSIS

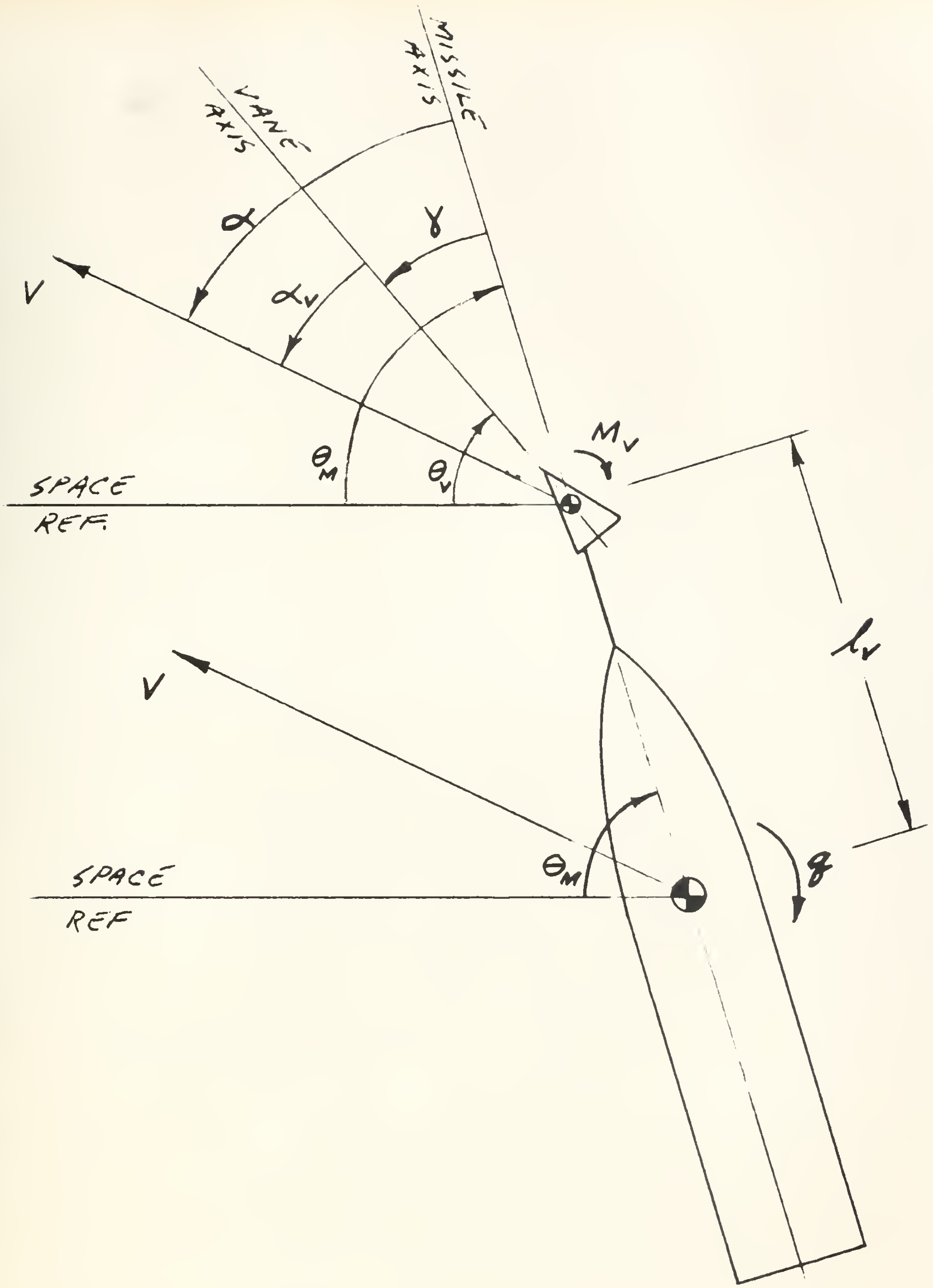


FIG. 11

SIGN CONVENTIONS FOR VANE ANALYSIS

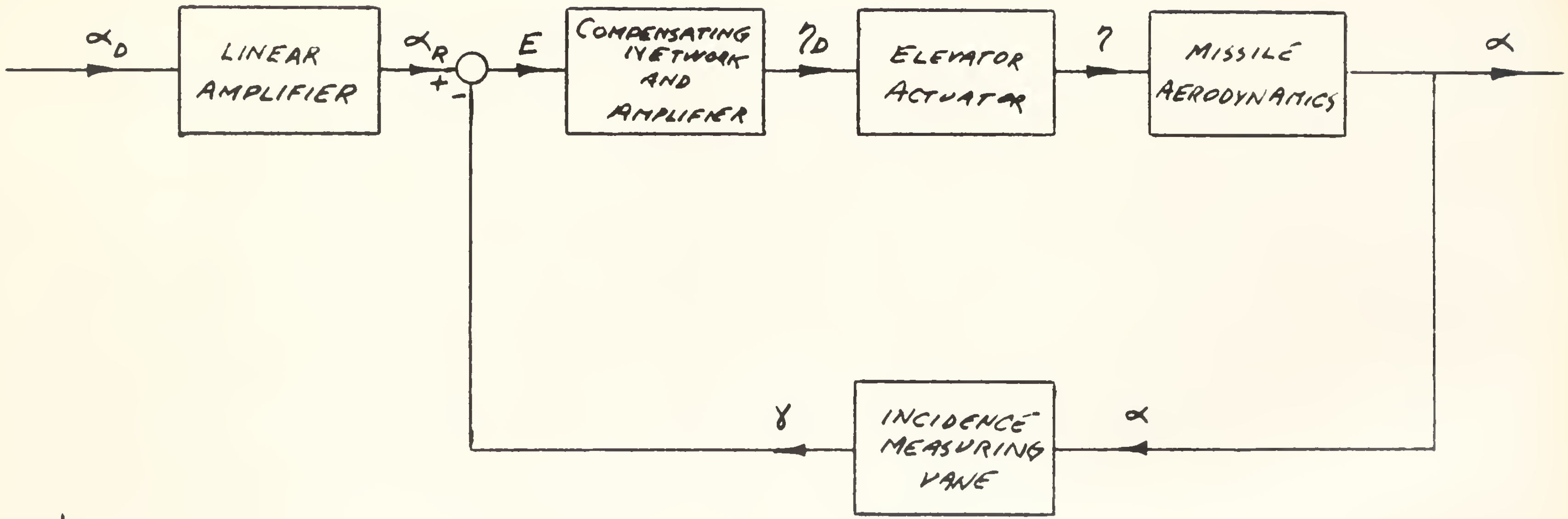
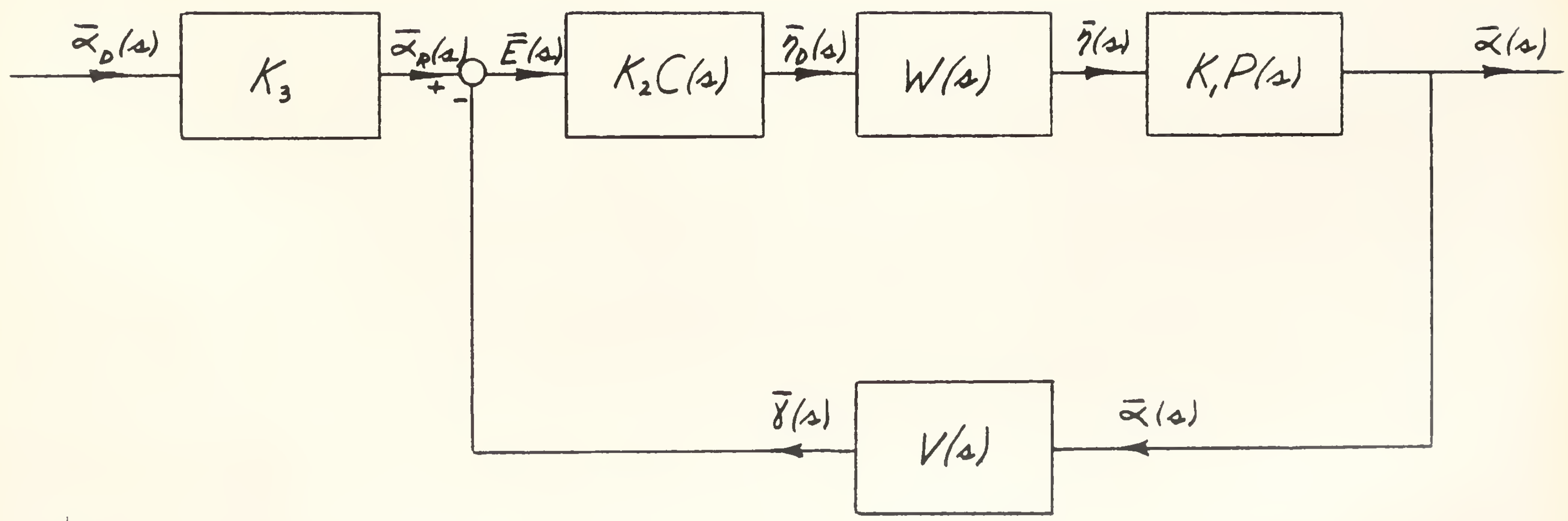


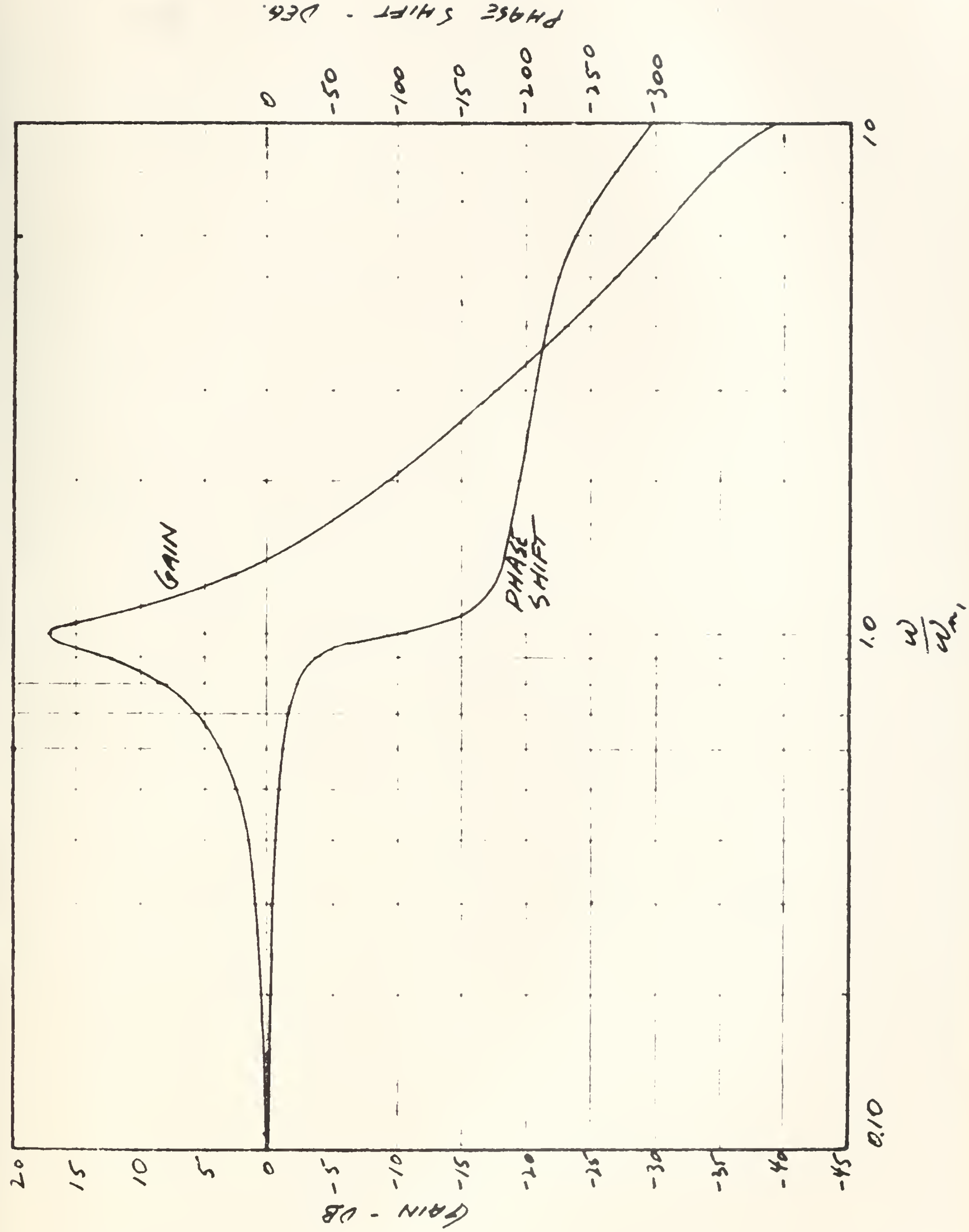
FIG. 12

CONTROL SYSTEM BLOCK DIAGRAM



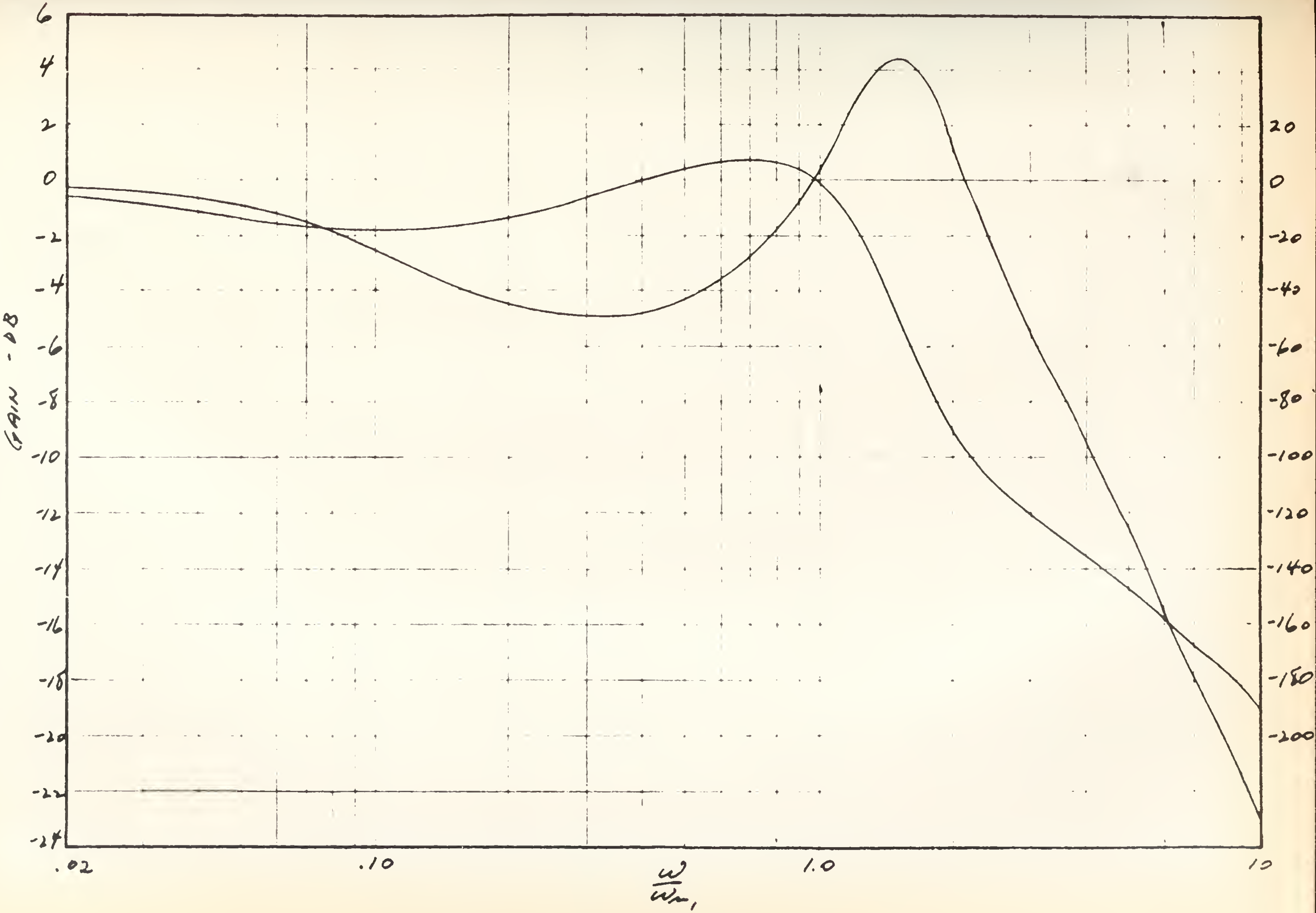
TRANSFORMED BLOCK DIAGRAM

FIG. 13



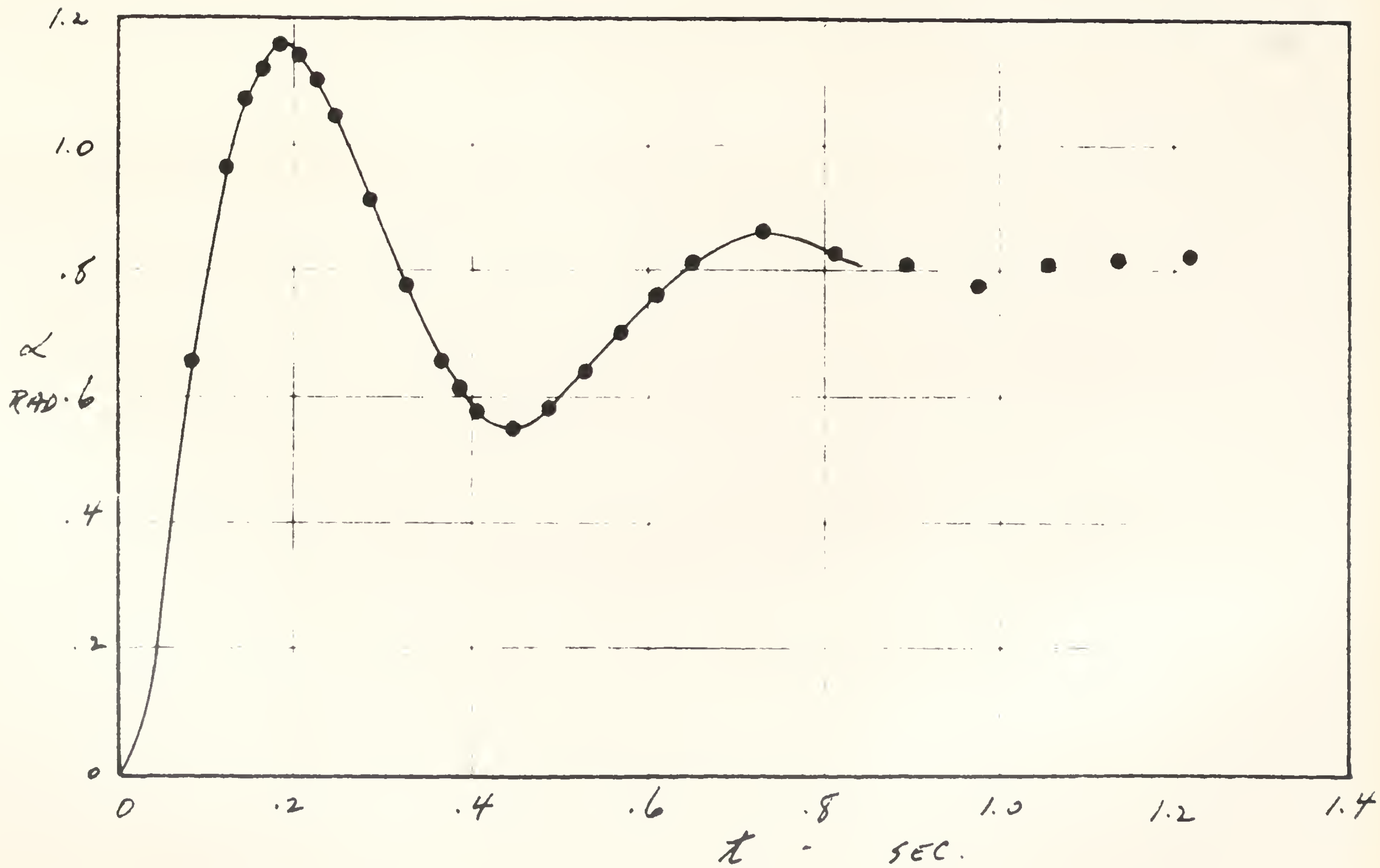
FIXED COMPONENTS FREQUENCY RESPONSE - $P(i\omega)V(i\omega)W(i\omega)$

FIG. 14



CLOSED LOOP FREQUENCY RESPONSE

FIG. 15



LINEAR SYSTEM - TRANSIENT RESPONSE

FIG. 16

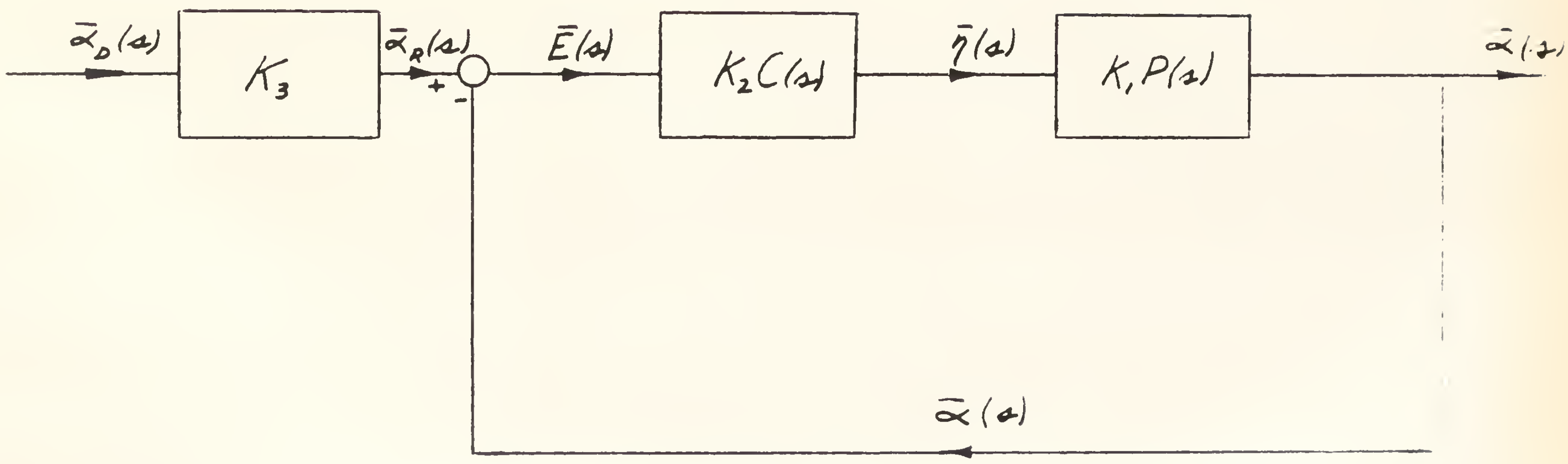


FIG. 17

SIMPLIFIED SYSTEM TRANSFORMED BLOCK DIAGRAM

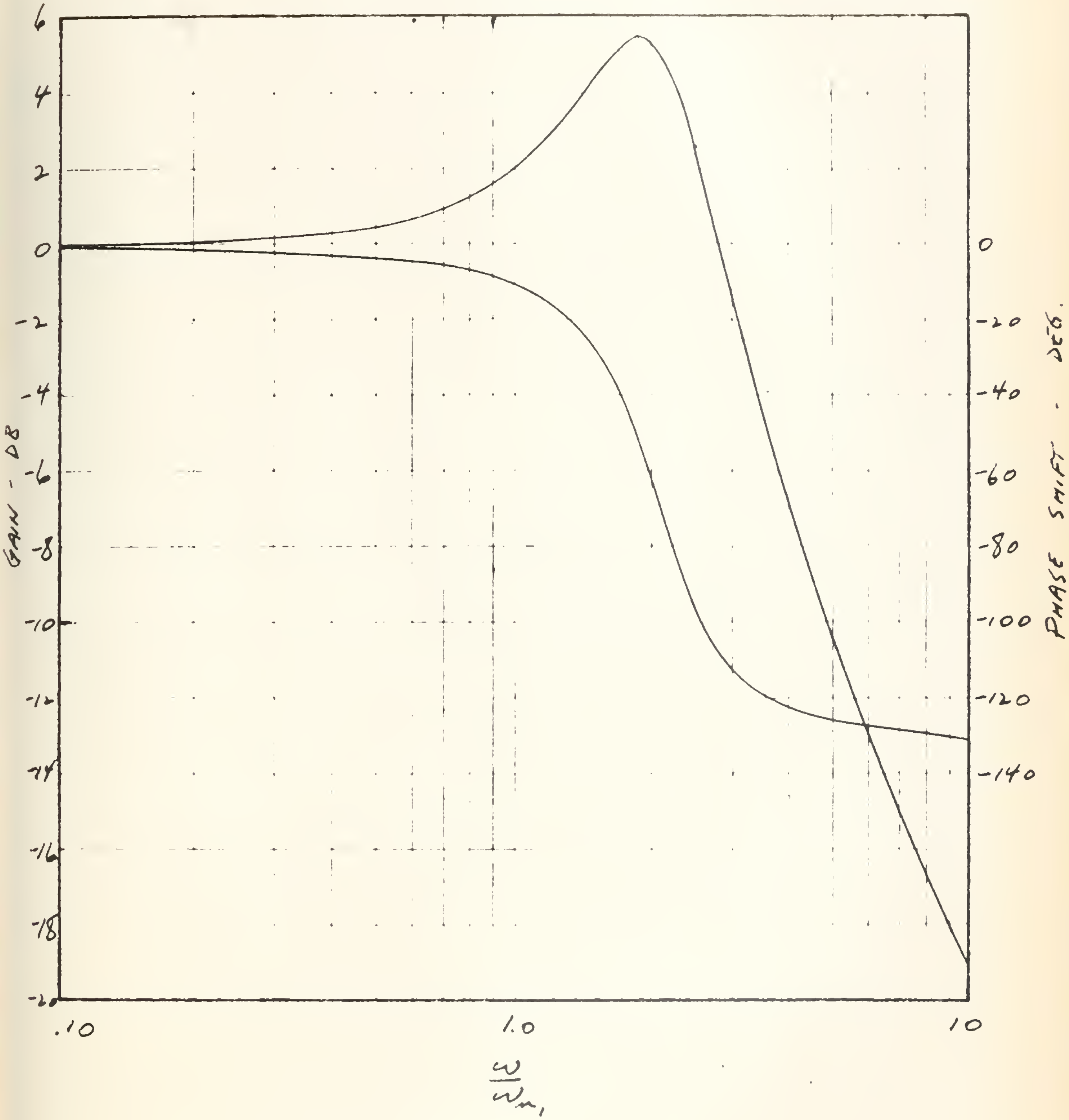


FIG. 18

SIMPLIFIED SYSTEM CLOSED-LOOP FREQ. RESP.

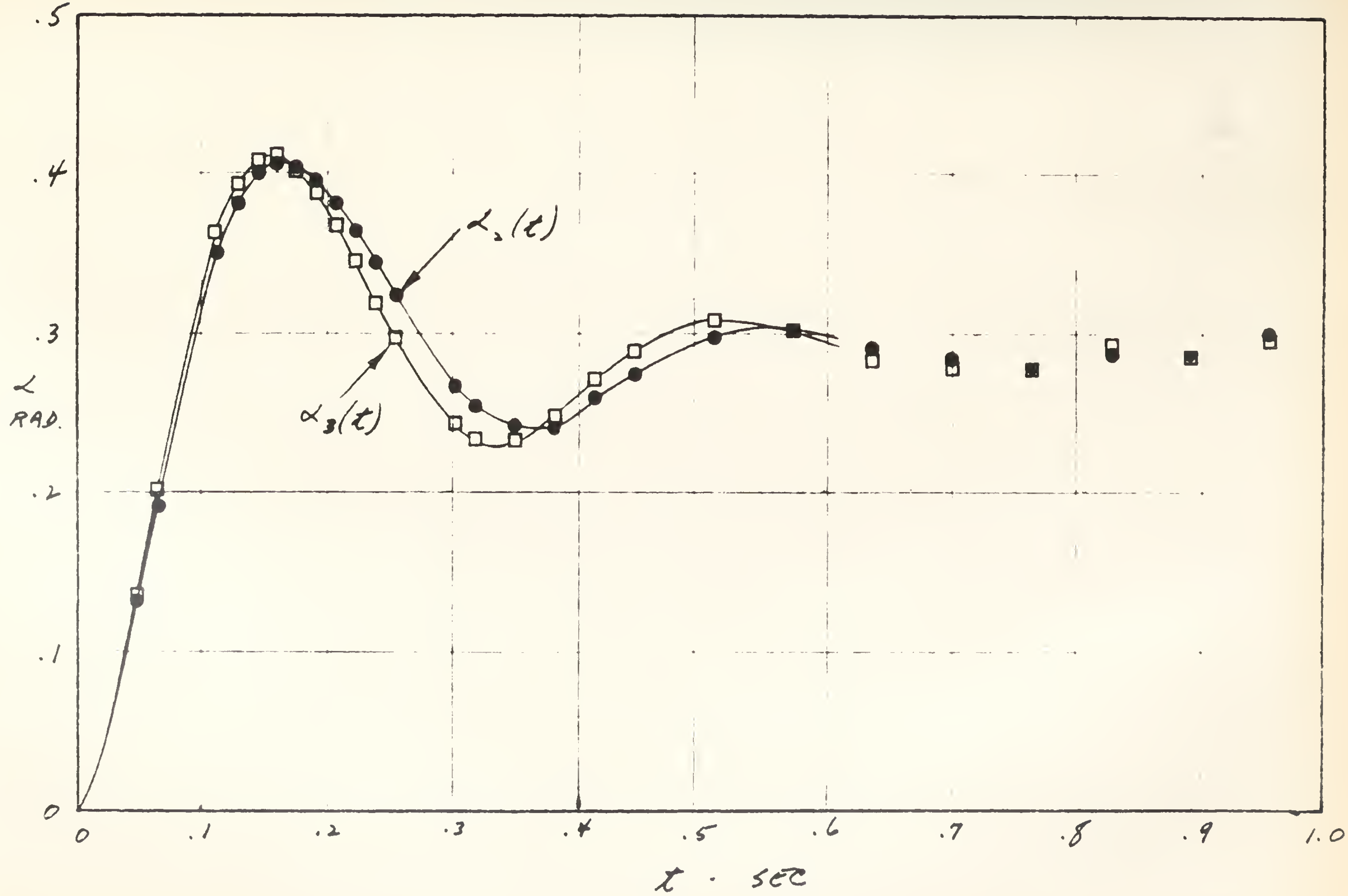
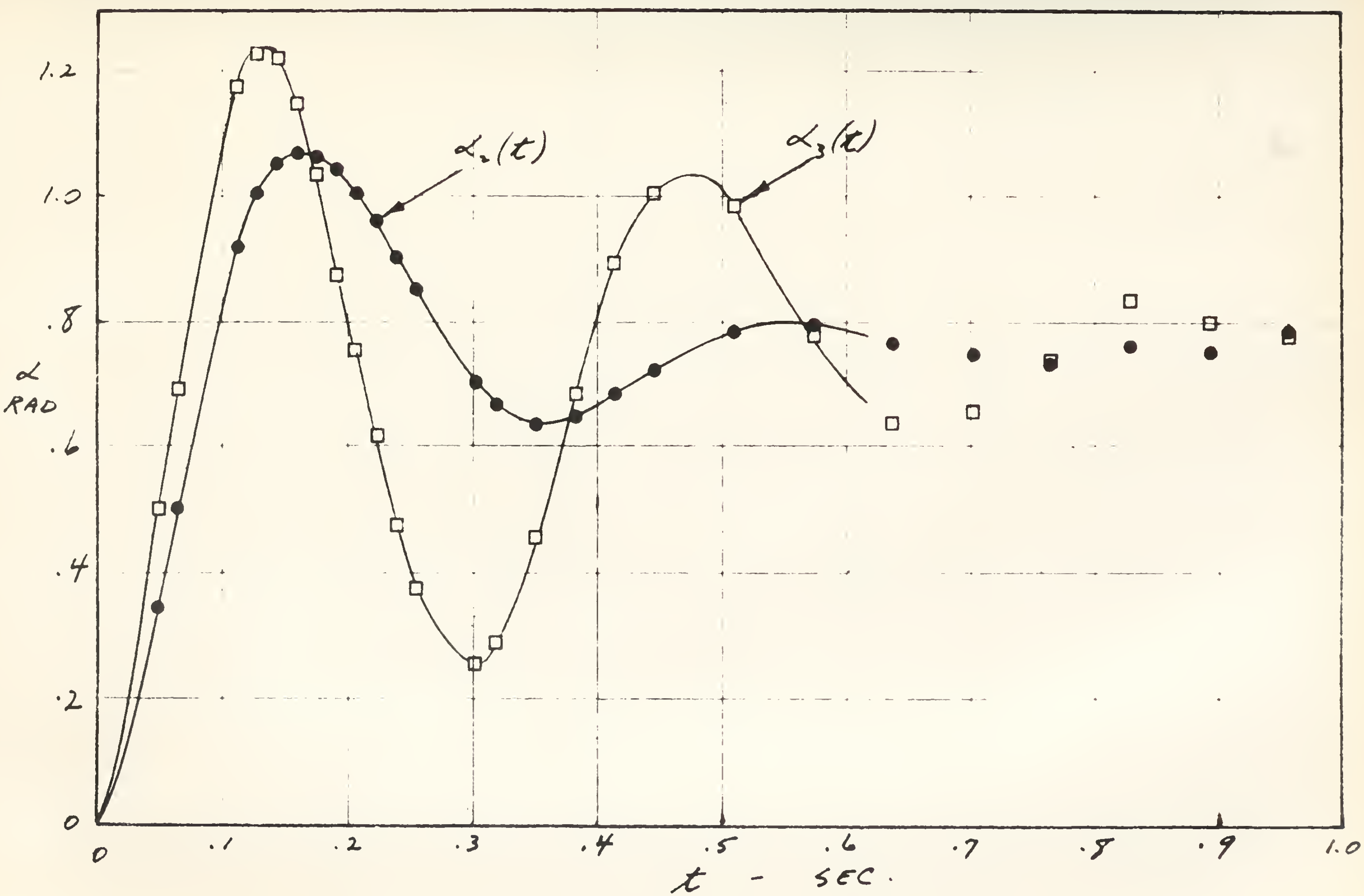


FIG. 19

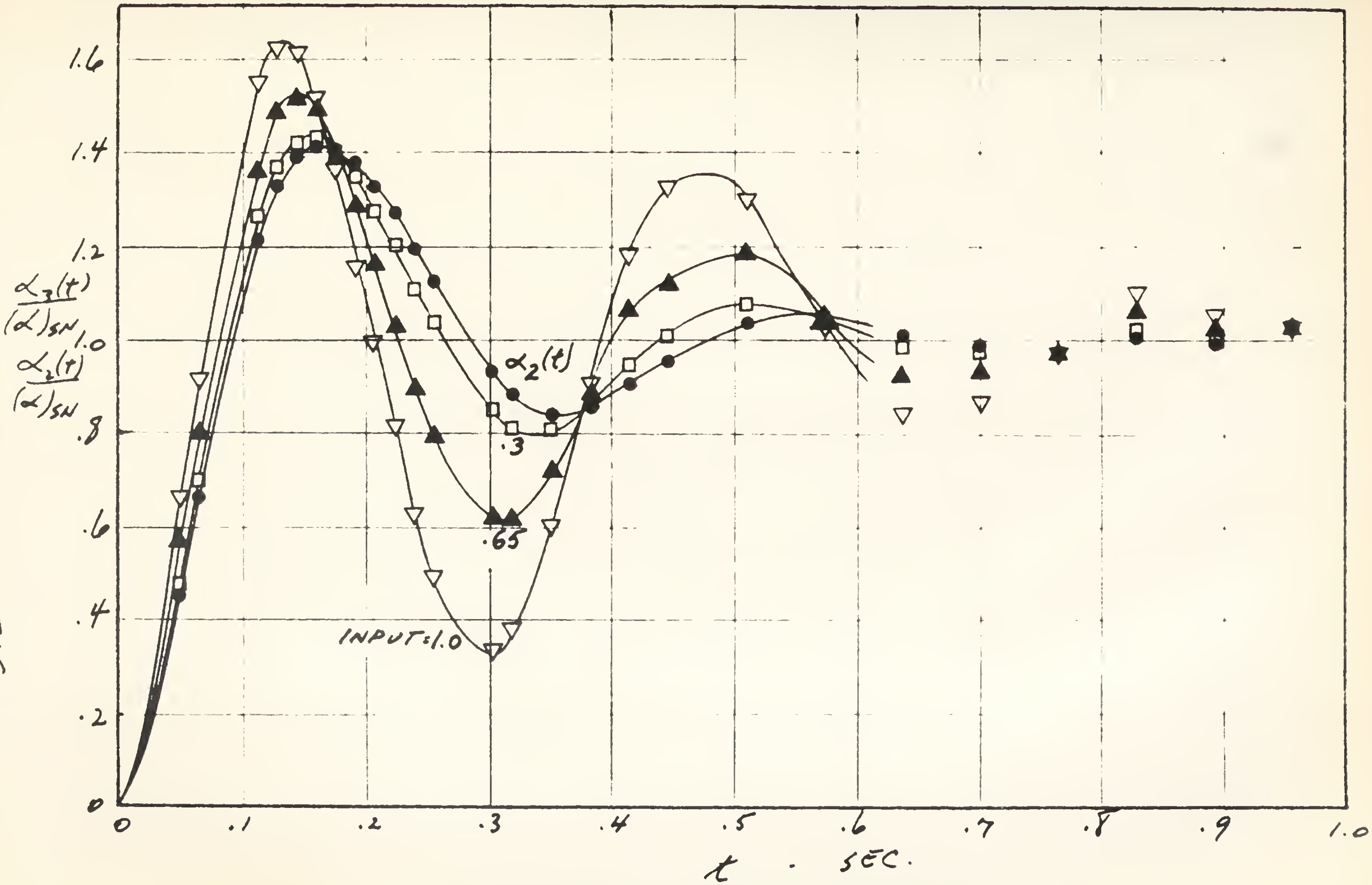
APPROXIMATE TRANSIENT RESPONSE OF NON-LINEAR SYSTEM - .3 INPUT



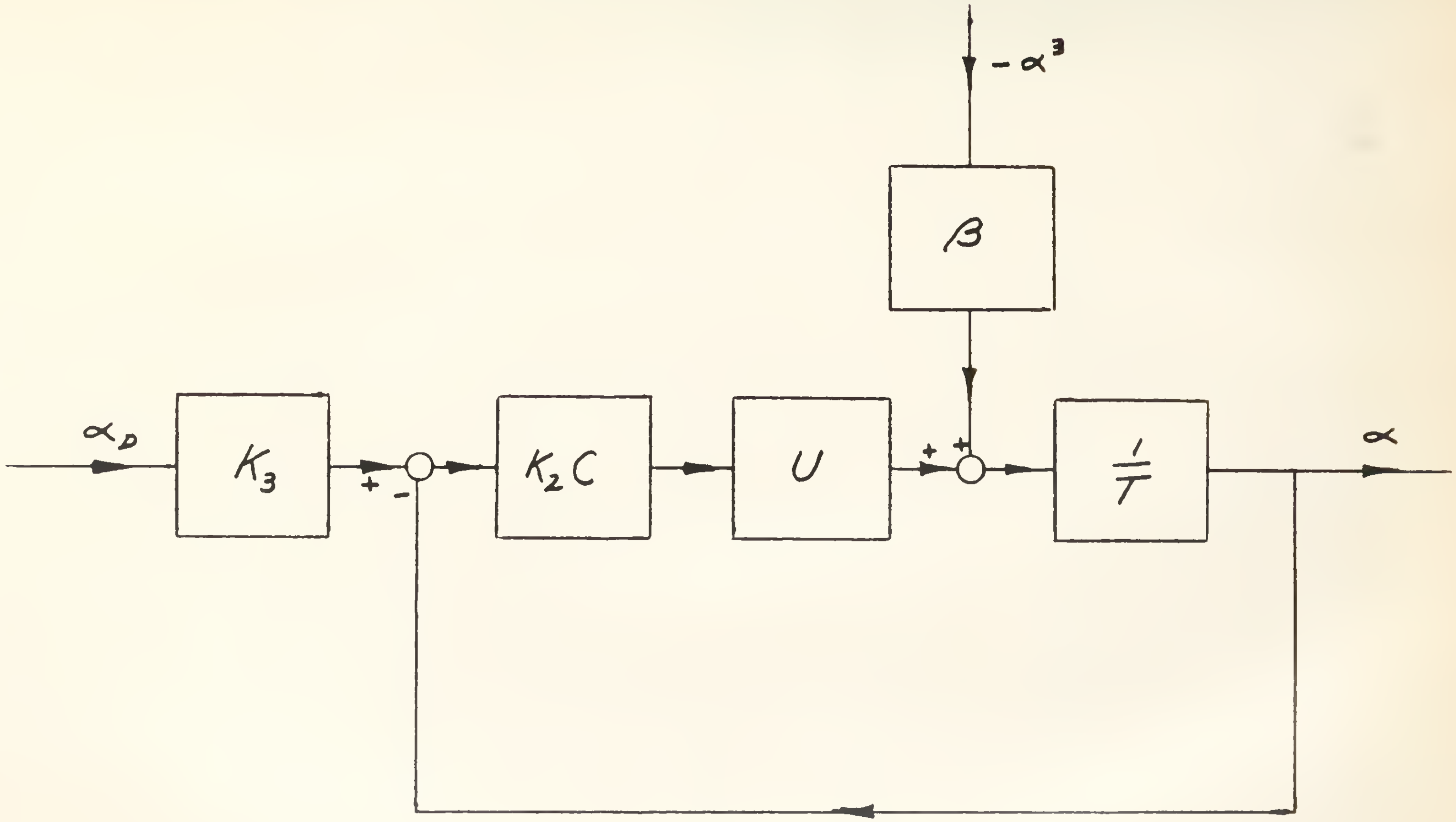
APPROXIMATE TRANSIENT RESPONSE OF NON-LINEAR SYSTEM - 1.0 INPUT.

FIG. 20

FIG. 21



NON-DIMENSIONAL APPROXIMATIONS TO NON-LINEAR TRANSIENT RESPONSE



EQUIVALENT LINEAR SYSTEM

FIG. 22

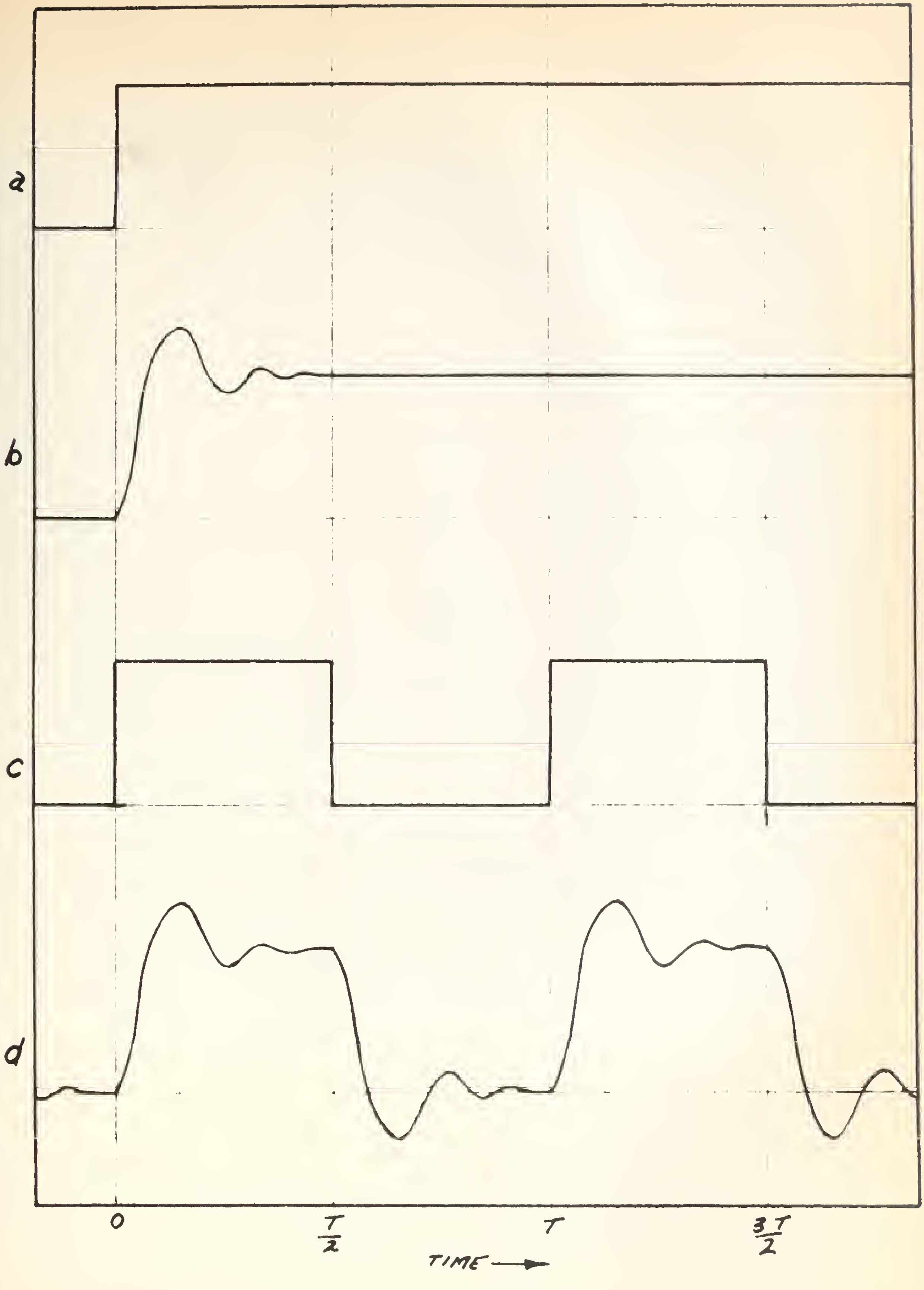


FIG. 23 - RESPONSE TO STEP AND TO SQUARE WAVE

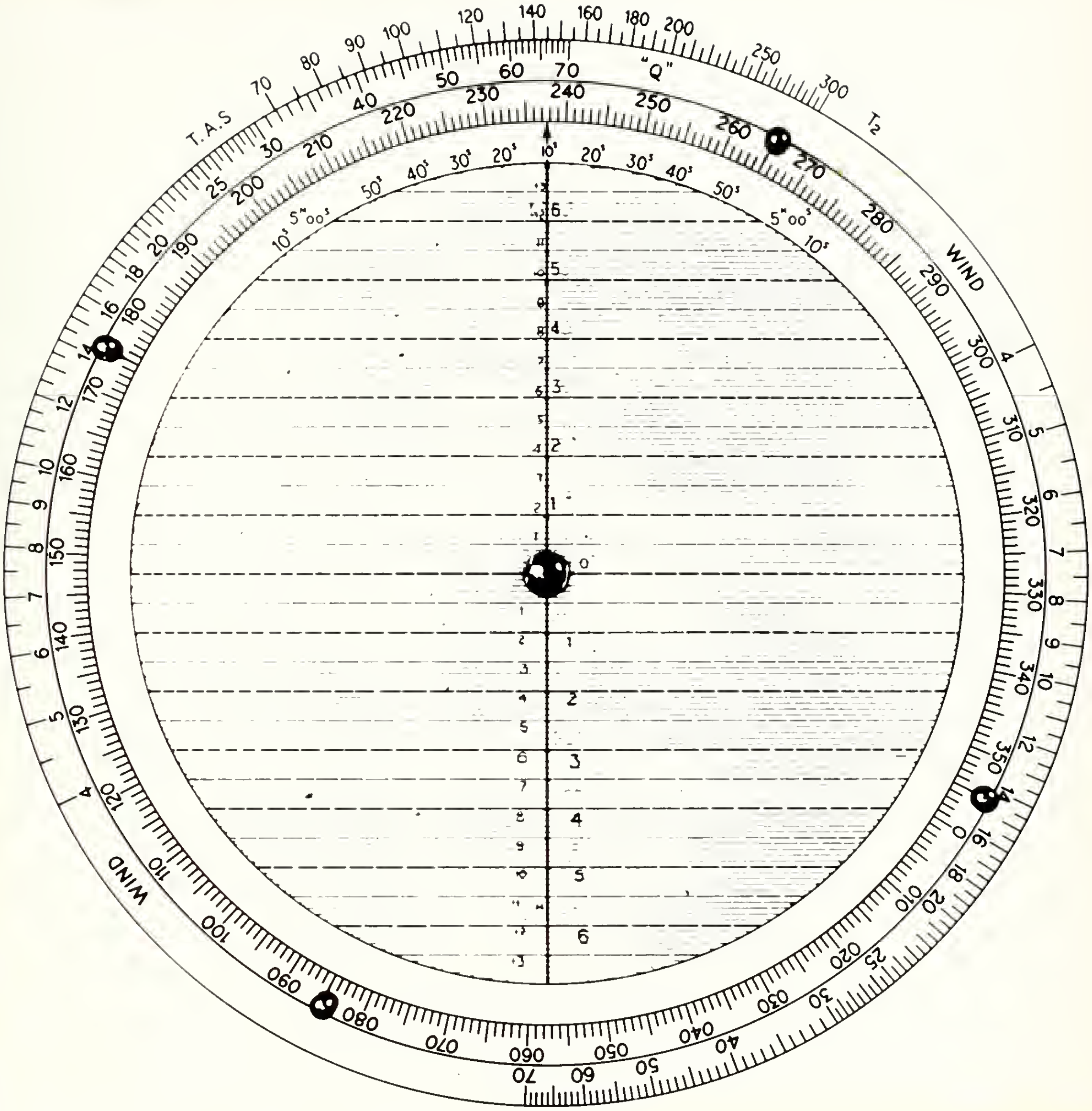


FIG. 24

CIRCULAR COMPUTER

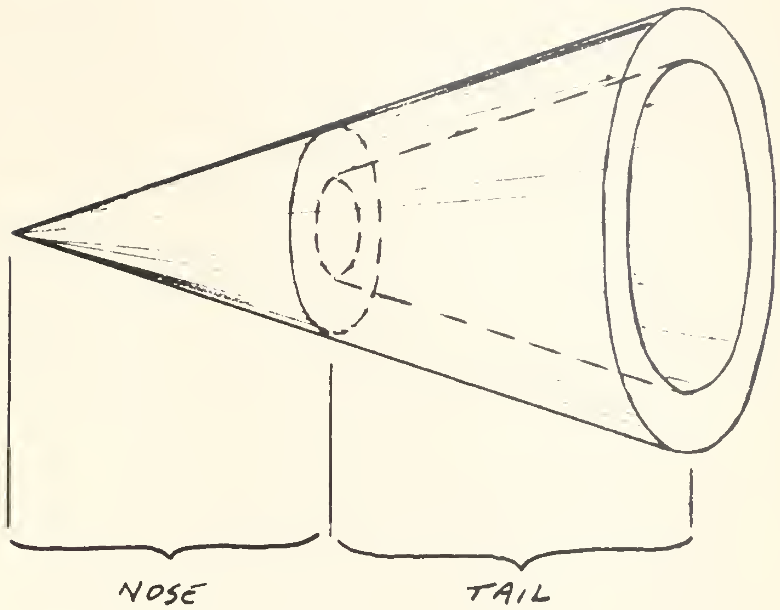


FIG. 25

INCIDENCE MEASURING VANE

APPENDIX I

CALCULATION OF THE AERODYNAMIC CONSTANTS

Brown-Edwards, in Ref. 1, found, for the missile under consideration and for a c.g. position 8 body diameters aft of the body apex, the following values:

$$C_L = 2.87\,\alpha - .222\,\eta \tag{I-1}$$

$$C_{M_{c.g.}} = -1.14\,\alpha - 2.65\,\alpha^3 + .97\,\eta \tag{I-2}$$

Fixing the body diameter l at one foot, he estimated for a typical missile:

$$m = 12.05 \ \frac{\text{lb.sec}^2}{\text{ft.}}$$

and

$$I = 230 \text{ lb.ft.sec.}^2$$

The reference area S is 9 ft^2.

We take the case of flight at 50,000' at Mach 3. The air density ρ is .000362 $\text{lb.sec}^2/\text{ft}^4$ and the speed of sound c is 968 ft/sec. Thus the velocity is:

$$V = 3\ (968) = 2900 \text{ ft/sec.}$$

The dynamic pressure is

$$Q = \tfrac{1}{2}\rho V^2 = \tfrac{1}{2}(.000362)(2900)^2 = 1520 \ \frac{\text{lb}}{\text{ft}^2}$$

and

$$QS = (1520)(9) = 13{,}700 \text{ lb.}$$

Now

$$Z_\alpha \approx \frac{\partial L}{\partial \alpha} = -Q\,S\,\frac{\partial C_L}{\partial \alpha} = -(13{,}700)(2.87) = -39{,}400 \text{ lb.}$$

where $\partial C_L/\partial \alpha$ is obtained from Eq. I-1.

$$Z_\eta \approx -\frac{\partial L}{\partial \eta} = -Q\,S\,\frac{\partial C_L}{\partial \eta} = -(13{,}700)(-.222) = 3{,}040 \text{ lb.}$$

where $\partial C_L/\partial \eta$ is obtained from Eq. I-1

$$M_\eta = \frac{\partial M}{\partial \eta} = QS\ell\,\frac{\partial C_{M_{c.g.}}}{\partial \eta} = (13{,}700)(1)(.97) = 13{,}300 \text{ lb.ft.}$$

where $\partial C_{M_{c.g.}}/\partial \eta$ is obtained from Eq. I-2. The moment due to α is, from Eq. I-2,

$$M''(\alpha) = QS\ell\,(-1.14\,\alpha - 2.65\,\alpha^3)$$

$$= (13{,}700)(1)(-1.14\,\alpha - 2.65\,\alpha^3)$$

$$= (-15{,}600\,\alpha - 36{,}300\,\alpha^3) \text{ lb.ft.}$$

Comparing this with Eq. 1-14, we see that

$$M_\alpha = -15{,}600 \text{ lb.ft.}$$

and

$$M_{\alpha^3} = -36{,}300 \text{ lb.ft.}$$

Using Eqs. 1-17 to 1-21,

$$\omega_{n_1}^{\,2} = -\frac{M_\alpha}{I} = \frac{15{,}600}{230} = 67.8 \text{ sec}^{-2}.$$

$$\omega_{n_1} = 8.23 \text{ sec}^{-1}$$

$$\zeta_1 = - \frac{Z_\alpha}{2 m V \omega_{n_1}} = \frac{39,400}{2(12.05)(2900)(8.23)}$$

$$= .0685$$

$$K_1 = \frac{M_\eta}{I \omega_{n_1}^2} = \frac{13,300}{(230)(67.8)} = .853$$

$$\tau = \frac{Z_\eta}{m V K_1 \omega_{n_1}^2} = \frac{3,040}{(12.05)(2900)(.853)(67.8)}$$

$$= .001505 \text{ sec.}$$

$$\beta = - \frac{M_{\alpha 3}}{I} = \frac{36,300}{230} = 158 \text{ sec}^{-2}$$

APPENDIX II

THE WASS, HAYMAN METHOD OF DERIVING THE TRANSIENT RESPONSE OF A LINEAR SYSTEM FROM THE FREQUENCY RESPONSE

Reference 2 presents a method of deriving the approximate response of a linear system to a step input from the frequency response. The method is to find the transient response to a square wave input instead of to a step input. The justification for such a manouver is best explained with reference to Fig. 23. Figure 23a is the graph of a step input, and Fig. 23b is the corresponding output of a typical system. Figure 23c is a square wave input of period T, and Fig. 23d is the corresponding output. It is evident that if the half-period T/2 is greater than the time required for the transient to completely damp out, then any of the half cycles of Fig. 23d is the same as the transient part of Fig. 23b. Of course in linear systems the transient never completely damps out. Therefore Fig. 23d is really an approximation to Fig. 23b, its accuracy depending on the extent to which the transient subsides in the time T/2.

The reason for using a square wave is that it can be expressed as a Fourier series. The Fourier series for a square wave $F_i(t)$ with equal mark-to-space ratio, minimum value zero, maximum value unity, and fundamental frequency ω_F, can be written:

$$F_i(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n = 1, 3, 5, \ldots, \infty} \frac{1}{n} \sin n\omega_F t$$

(Wass and Hayman use the symbol n differently - this will be explained shortly). If $F_i(t)$ is the input to a linear system the magnitude and phase of each component will be changed according to the frequency response of the system, and the

output will be

$$F_o(t) = \frac{A_o}{2} + \frac{2}{\pi} \sum \frac{A_n}{n} \sin(n\omega_F t + B_n)$$

$$n = 1, 3, 5, \ldots \infty$$

where A_n is the gain of the system at frequency $n\omega_F$,

B_n is the phase shift of the system at frequency $n\omega_F$.

In order to use this method numerically, the summation must be limited to a reasonably low number of terms. Wass and Hayman state that it has been found by experience that a satisfactory compromise between labor and accuracy is achieved by considering up to the 11th, or sometimes 13th, harmonic of the fundamental square wave frequency. Having decided on the number of terms to be summed, a value for the fundamental frequency ω_F must be chosen. Wass and Hayman suggest that for systems whose gain curves exhibit a definite peak, ω_F should be one fifth of the frequency of the peak. Suggestions are also made with respect to other types of systems.

Wass and Hayman do not point out that taking the summations up to even the 13th harmonic will fail to give satisfactory accuracy in certain types of systems. They do infer that the transient response is mainly determined by the characteristics of the frequency response at frequencies lower than that at which the gain is 15 db below the zero frequency gain. They do not point out that the gain and phase shift at the fundamental frequency must not differ greatly from their zero frequency values. For conciseness we may use the term "frequency spread" to denote the ratio of the -15 db frequency to the frequency at which the gain or phase first differs appreciably from its zero frequency value. Thus the Wass, Hayman method, summing up to the 13th

harmonic, is not applicable to systems whose frequency spread is greater than 13.

An important part of the Wass, Hayman report is a table, the "φ" Table, which may be used to simplify the evaluation of Fourier series. This is a table giving the values of $n\omega_F t'$ in degrees, for 39 values of non-dimensional time t', and for odd values of n from 1 to 13. Even values of n are not included because the Fourier series for the responses of linear systems to square wave inputs contain only odd harmonics. Since the evaluation of even harmonics is required in this thesis, an extension to the φ Table is required, and this is given, for even values of n from 2 to 8, in Table XV. Wass and Hayman's use of the symbol n precludes the possibility of even harmonics, and this is the reason it is used differently here.

Wass and Hayman present a design for a circular computer which reduces considerably the time required to evaluate a Fourier series. The computer suggested consists of a circular base with various lines and scales marked on it, two arms which pivot at the center of the circular base, and a cursor which slides along one of the arms. The author constructed and used such a computer, and eventually devised a different design which was superior in several respects. Fig. 24 is a photograph of this computer. It was originally a "Wind and Navigation Calculator Mk. I", and all the scales around the periphery are superfluous except the 360 degree one. It consists of an opaque base and a transparent top, pivoted at the center. The horizontal lines, the numbers running down the center, and the arrowhead are etched on the base. The 360° scale is etched on the top. The top is roughened so that it will take pencil marks. To evaluate the term

$$A = B \sin (n\omega_F t' + \theta)$$

the procedure is as follows:

1. Rotate the top until the arrowhead points to $(90^\circ - \theta)$ on the 360° scale.

2. Make a pencil mark, on the top, at the point where the value of B appears on one of the vertical scales above the center.

3. Rotate the top until the arrowhead points to $n\omega_F t'$ on the 360° scale.

4. Follow the new position of the pencil mark along the horizontal lines to the same scale used to place the mark, and read A. It is positive if above center, negative if below.

5. Repeat steps 3 and 4 for other values of $n\omega_F t'$.

The new computer is superior to the Wass, Hayman version in these respects:

a. The two arms of the Wass, Hayman computer must be turned through several revolutions without moving relative to each other, yet the angle between them must be adjustable. A design incorporating such a feature may be difficult to achieve, particularly for the home builder. The new computer obviates this difficulty.

b. The angles between the arms and the position of the cursor must be reset for each new term of the Fourier series. With the new computer a mark for each term of the series can be made at the outset, each one being labeled appropriately. This may not seem important, but in practice it is. Mistakes can be corrected easily because the setting is not "erased" for the next term as with the Wass, Hayman version, and evaluations can be made for additional values of time, without resetting, after several points have been plotted.

APPENDIX III

GRAPHICAL ADDITION OF LIKE HARMONICS

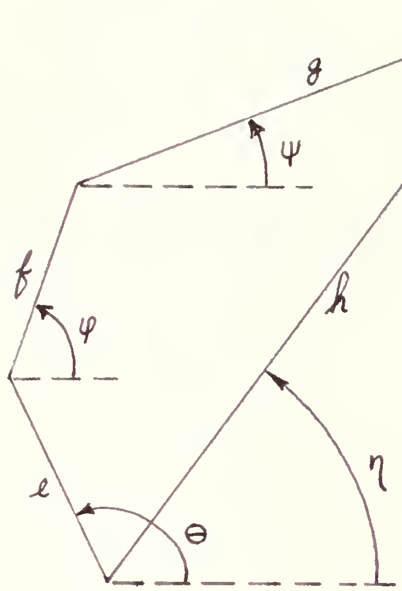
If, in the expression

$$e \sin(\omega t + \theta) + f \sin(\omega t + \varphi) + g \sin(\omega t + \psi)$$

e, f, g, θ, φ, and ψ are known constants, then an equivalent expression is

$$h \sin(\omega t + \eta)$$

where h and η are determined according to the construction below.



This method is valid for any number of terms in the original expression.

APPENDIX IV

TRANSFORMATION OF THE CUBIC EQUATION

Given the equation:

$$\frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_o x + b x^3 = Q \qquad \text{(I-1)}$$

Make the substitutions:

$$t = \frac{1}{\sqrt{a_o}} \tau$$

and

$$x = \sqrt{\frac{a_o}{b}} \, y$$

Then Eq. I-1 becomes:

$$\frac{d^2y}{d\tau^2} + \frac{a_1}{\sqrt{a_o}} \frac{dy}{d\tau} + y + y^3 = \frac{Q\sqrt{b}}{a_o^{3/2}}$$

We shall call the quantity

$$Q_o \equiv \frac{Q\sqrt{b}}{a_o^{3/2}}$$

the non-dimensional input. The coefficient $\frac{a}{\sqrt{a_o}}$ is equal to twice the damping ratio ζ.

APPENDIX V

DESIGN OF THE INCIDENCE VANE

The vane is conical, and it is necessary that the c.g. be as far forward as possible. The configuration to be used is that shown in Fig. 25. The nose will be solid steel, and the base will be a thin shell of light-weight plastic. The optimum ratio of nose length to tail length, for most forward c.g. position, is that which places the c.g. at the junction of the nose and tail. The equation for this condition was found to be:

$$\frac{t}{L} = \frac{1}{4}\frac{\rho_1}{\rho_2}\frac{\sin\theta}{(2n^4 - 3n^3 + n)}$$

where
- t is the base thickness,
- L is the cone length,
- ρ_1 is the density of the nose material,
- ρ_2 is the density of the base material,
- θ is the cone semi-angle,
- $\frac{L}{n}$ is the distance of the c.g. from the apex.

The density of steel is about .3 lb/cu in., and that of a high-temperature silicone plastic is about .05 lb/cu in. (p. 66 of Ref. 4). Using these values, the compromise chosen was:

$$\theta = 22.5^\circ$$
$$n = 3$$
$$L = 3''$$
$$t = .0614''$$

It is questionable whether the base is thick enough for adequate strength, but it will be assumed that such a design is practicable.

APPENDIX VI

CALCULATION OF THE VANE CONSTANTS

Ref. 5 gives experimental pitching moment data for cones in supersonic flow in the form of graphs of C_M versus α, where the pitching moment about the nose is

$$M_{NOSE} = \pi Q r_b^2 l_c C_M$$

and

$$r_b = \text{base radius,}$$

$$l_c = \text{cone length.}$$

The curves of C_M versus α were virtually linear for the conditions applicable to the present case, and the value of $\partial C_M / \partial \alpha$ about the nose was .020. The center of pressure remained very close to a position 2/3 l_c back from the nose, so that the moment curve slope for a cone hinged at 1/3 l_c would be half of .020, or .010.

From Appendix I:

$$Q = 1520 \text{ lb/ft}^2$$

From Appendix V

$$l_c = .25 \text{ ft.}$$

$$r_b^2 = .0107 \text{ ft}^2.$$

Thus:

$$C_1 = -\left(\frac{\partial M_V}{\partial \alpha}\right)_{\ell/3} = -\pi Q r_b^2 \; l_c \left(\frac{\partial C_M}{\partial \alpha}\right)_{1/3}$$

$$= -\pi(1520)(.0107)(.25)(.01)$$

$$= - \; .128 \text{ ft.lb.}$$

$$I_V = 1.70 \times 10^{-5} \text{ ft.lb.sec}^2.$$

(Found by integration, calculation not shown).

$$\omega_{n_2}^{\ 2} = -\frac{C_1}{I_V} = \frac{.128}{1.70 \times 10^{-5}} = 7{,}530 \text{ sec}^{-2}$$

$$\omega_{n_2} = 86.9 \text{ sec}^{-1}$$

$$\zeta_2 = .5$$

(Chosen arbitrarily on the assumption that the vane damping device can be so designed.)

$$C_2 = 2\, I_V\, \omega_{n_2} \zeta_2 = 2(1.70 \times 10^{-5})(86.9)(.5)$$

$$= .00148 \text{ ft.lb.sec.}$$